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Projections in L^1-algebras and tight frames

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ABSTRACT. Any invertible $n \times n$-matrix acting on $\mathbb{R}^n$ admits a tight frame generator if its determinant is not 1 or -1. However, a tight frame generator with some degree of smoothness exists if and only if the matrix or its inverse is expansive. These latter properties are also shown to be equivalent to representation theoretic properties of a naturally associated semi-direct product group.

1. Introduction

In this note we provide an investigation of the connections between projections in $L^1(G)$, for a locally compact group G and the existence of tight frames with some degree of smoothness. To keep the discussion focussed and concise, we will study an explicit class of examples; semi-direct product groups formed by the action of the integers $\mathbb{Z}$ on $\mathbb{R}^n$ through the powers of an invertible matrix A and tight frames associated with A as a dilation matrix. In section 2, we present our main theorem which establishes direct connections.

We begin by introducing a general notion of a tight frame following the spirit of [**1**], where even more general concepts are studied. Often frames in Hilbert space are considered as discretely parametrized, but it is useful, and easy, to expand the concept to include parametrization by a measure space.

Let $\mathcal{H}$ be a Hilbert space, (Ω, m) a measure space and $\omega \to \eta_\omega$ a measurable map of Ω into $\mathcal{H}$. We call $\{\eta_\omega : \omega \in \Omega\}$ a (normalized) tight frame in $\mathcal{H}$ if, for any $\xi \in \mathcal{H}$,

$$\|\xi\|^2 = \int_\Omega |\langle \xi, \eta_\omega \rangle|^2 dm(\omega). \tag{1.1}$$

When $\{\eta_\omega : \omega \in \Omega\}$ is a tight frame in $\mathcal{H}$, any vector $\xi \in \mathcal{H}$ can be recovered from the coefficients, $\{\langle \xi, \eta_\omega \rangle : \omega \in \Omega\}$, via

$$\xi = \int_\Omega \langle \xi, \eta_\omega \rangle \eta_\omega dm(\omega), \tag{1.2}$$

with weak convergence of the integral (see [**1**]).

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For a locally compact group G, denote the left Haar integral of a function f on G by $\int_G f(x)dx$. Let $L^1(G)$ denote the usual Banach space of absolutely integrable complex-valued functions on G. For $f, g \in L^1(G)$, the convolution $f * g$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \tag{1.3}$$

for all $x \in G$. With convolution as product, $L^1(G)$ is a Banach algebra. Moreover, $L^1(G)$ carries the natural involution $f \to f^*$, where $f^*(x) = \Delta(x^{-1})\bar{f}(x^{-1})$, for all $x \in G$. Here Δ denotes the modular function of G. We refer to [**4**] for these and other basic facts of harmonic analysis on locally compact groups.

Any continuous unitary representation π of G can be integrated to form a $*$-representation of $L^1(G)$. That is, for $f \in L^1(G)$,

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx, \tag{1.4}$$

for all $\xi, \eta \in \mathcal{H}_\pi$.

Let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G, equipped with the Mackey-Fell topology (see [**3**], 18.1).

A function $f \in L^1(G)$ is called a projection if f is a self-adjoint idempotent ($f = f^* = f * f$). If $\mathcal{E}$ is a faithful family of $*$-representations of $L^1(G)$ and $f \in L^1(G)$, then f is a projection if and only if $\pi(f)$ is an orthogonal projection on the Hilbert space of π, for all $\pi \in \mathcal{E}$. If f is a projection in $L^1(G)$, let

$$s(f) = \{\pi \in \hat{G} : \pi(f) \neq 0\}. \tag{1.5}$$

It follows from [**3**], 3.3.2 and 3.3.7, that $s(f)$ is a compact open subset of $\hat{G}$. If f is a non-zero projection, then $s(f)$ is non-empty. (Note that $\hat{G}$ need not be T_2, so compact subsets need not be closed.) Thus, if G is a group for which $\hat{G}$ has no non-empty compact open subsets, then there are no non-zero projections in $L^1(G)$. This fact was used in [**5**] and [**6**] to study the existence and construction of projections in certain semi-direct product groups.

We now define the class of groups that is our main focus in this article. This class is relevant to the theory of tight frames and wavelet analysis as groups in the class are composed of dilations and translations of $\mathbb{R}^n$. It is also close enough to the class of groups studied in [**5**], that the techniques of constructing projections, when they exist, can be adapted from [**5**].

Let n be a positive integer and let $A \in GL_n(\mathbb{R})$, the group of invertible $n \times n$ real matrices. We consider $\mathbb{R}^n$ as consisting of column vectors. For $k \in \mathbb{Z}$ and $\boldsymbol{x} \in \mathbb{R}^n$, let $[k, \boldsymbol{x}]$ denote the affine transformation given by

$$[k, \boldsymbol{x}]\boldsymbol{z} = A^k(\boldsymbol{x} + \boldsymbol{z}), \tag{1.6}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$. Calculating the composition of $[k, \boldsymbol{x}]$ and $[\ell, \boldsymbol{y}]$ gives

$$[k, \boldsymbol{x}][\ell, \boldsymbol{y}] = [k + \ell, A^{-\ell}\boldsymbol{x} + \boldsymbol{y}]. \tag{1.7}$$

This defines a group product on $\mathbb{Z} \times \mathbb{R}^n$ with the group inverse given by $[k, \boldsymbol{x}]^{-1} = [-k, -A^k\boldsymbol{x}]$. The resulting group is actually a semi-direct product of $\mathbb{Z}$ with $\mathbb{R}^n$, which we will denote as G_A to emphasize how much its nature depends on the matrix A.

There is a natural unitary representation ρ of G_A on the Hilbert space $L^2(\mathbb{R}^n)$ that arises from the defining action of G_A on $\mathbb{R}^n$ as affine transformations. For $g \in L^2(\mathbb{R}^n)$ and $[k, \boldsymbol{x}] \in G_A$, let

$$\begin{aligned} \rho[(k,\boldsymbol{x}])g(\boldsymbol{z}) &= \delta^{-k/2} g([k,\boldsymbol{x}]^{-1}\boldsymbol{z}) \\ &= \delta^{-k/2} g(A^{-k}\boldsymbol{z} - \boldsymbol{x}), \end{aligned} \tag{1.8}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$, where $\delta = |\det A|$. The factor of $\delta^{-k/2}$ in (1.8) is required to make $\rho[(k, \boldsymbol{x}])$ unitary.

For a fixed $w \in L^2(\mathbb{R}^n)$, we use ρ to move w around. Let $w_{k,\boldsymbol{x}}$, for $[k, \boldsymbol{x}] \in G_A$, be defined by

$$w_{k,\boldsymbol{x}} = \rho([k,\boldsymbol{x}])w. \tag{1.9}$$

Many of the fundamental questions in wavelet analysis involve the existence, and construction, of w such that $\{w_{k,\boldsymbol{x}} : k \in \mathbb{Z}, \boldsymbol{x} \in \Gamma\}$, for an appropriate subset Γ of $\mathbb{R}^n$, forms a tight frame (or orthonormal basis) in $L^2(\mathbb{R}^n)$ under various conditions on the matrix A. Here, we restrict our attention to $\Gamma = \mathbb{R}^n$ and the existence of a w such that $\{w_{k,\boldsymbol{x}} : k \in \mathbb{Z}, \boldsymbol{x} \in \mathbb{R}^n\}$ forms a tight frame. We use the product of counting measure on $\mathbb{Z}$ and Lebesgue measure on $\mathbb{R}^n$ as the product measure on the parameter space $\mathbb{Z} \times \mathbb{R}^n$. This happens to coincide with left Haar measure on G_A.

Definition 1.1. We will call $w \in L^2(\mathbb{R}^n)$ a tight frame generator for A (TFG$_A$) if

$$\|g\|_2^2 = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |\langle g, w_{k,\boldsymbol{x}}\rangle|^2 d\boldsymbol{x}, \tag{1.10}$$

for all $g \in L^2(\mathbb{R}^n)$.

The easiest way to check that (1.10) holds for a given w is to go to the Fourier transform. It is convenient to let $\hat{\mathbb{R}}^n$ denote $\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}\}$ and, for $g \in L^1(\mathbb{R}^n)$,

$$\hat{g}(\gamma) = \int_{\mathbb{R}^n} g(\boldsymbol{y}) e^{-2\pi i \gamma \boldsymbol{y}} d\boldsymbol{y}, \tag{1.11}$$

for all $\gamma \in \hat{\mathbb{R}}^n$. Then $g \to \hat{g}$ can be extended to a Hilbert space isomorphism of $L^2(\mathbb{R}^n)$ onto $L^2(\hat{\mathbb{R}}^n)$. A direct calculation, using the standard properties of the Fourier transform, shows that, for $w, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\langle g, w_{k,\boldsymbol{x}}\rangle|^2 d\boldsymbol{x} = \int_{\hat{\mathbb{R}}^n} |\hat{g}(\gamma)|^2 |\hat{w}(\gamma A^k)|^2 d\gamma \tag{1.12}$$

Summing (1.12) over $k \in \mathbb{Z}$ and noting that (1.10) must hold for all $g \in L^2(\mathbb{R})$ in order that w be a TFG$_A$, we get a variation on one of the common conditions in the theory of frames and wavelets.

Proposition 1.2. Let $A \in GL_n(\mathbb{R})$ and $w \in L^2(\mathbb{R}^n)$. Then w is a TFG$_A$ if and only if

$$\sum_{k=-\infty}^{\infty} |\hat{w}(\gamma A^k)|^2 = 1, \tag{1.13}$$

for almost every $\gamma \in \hat{\mathbb{R}}^n$.

For $\gamma \in \hat{\mathbb{R}}^n$, the A-orbit of γ is $\{\gamma A^k : k \in \mathbb{Z}\}$. For a TFG$_A$ w, the function $k \to \hat{w}(\gamma A^k)$ is a unit vector in $\ell^2(\mathbb{Z})$ for almost all γ. This is the source of the importance played by the structure of the A-orbits in $\hat{\mathbb{R}}^n$. Using that structure, we showed in [8] that a TFG$_A$ exists in $L^2(\mathbb{R}^n)$ if and only if $\delta \neq 1$. A much more general result on the existence of tight frame generators has since appeared in [7]. On the other hand, condition (1.13) also appears in [5] and [6] as part of the definition of a projection generating function. We will specialize the definition in [6] to G_A, using the realization of semi-direct products as in this note. To simplify the formulation, recall that $\mathcal{A}(\hat{\mathbb{R}}^n)$ denotes the Fourier algebra of $\hat{\mathbb{R}}^n$ and for, $\eta \in \mathcal{A}(\hat{\mathbb{R}}^n)$, $\eta^\vee$ denotes the $g \in L^1(\mathbb{R}^n)$ such that $\hat{g} = \eta$ (then $\|\eta\|_{\mathcal{A}(\hat{\mathbb{R}}^n)} = \|g\|_1$). Moreover, for a function $\xi : \hat{\mathbb{R}}^n \to \mathbb{C}$, let $\xi_k(\gamma) = \xi(\gamma A^k)$, for all $\gamma \in \hat{\mathbb{R}}^n$, $k \in \mathbb{Z}$.

DEFINITION 1.3. A function $\xi : \hat{\mathbb{R}}^n \to \mathbb{C}$ is called a projection generating function relative to the matrix A (PGF$_A$) if it has the following properties:

(i) The pointwise product $\xi_k \bar{\xi} \in \mathcal{A}(\hat{\mathbb{R}}^n)$, for all $k \in \mathbb{Z}$.

(ii) $\displaystyle\sum_{k=-\infty}^{\infty} \delta^{k/2} \|\xi_k \bar{\xi}\|_{\mathcal{A}(\hat{\mathbb{R}}^n)} < \infty$.

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} |\xi(\gamma A^k)|^2 = 1$, for all $\gamma \in \hat{\mathbb{R}}^n \setminus \{\mathbf{0}\}$

If ξ is a PGF$_A$, then define a function f_ξ on G_A by

$$f_\xi([k, \boldsymbol{x}]) = \delta^{-k/2} (\xi_{-k} \bar{\xi})^\vee (\boldsymbol{x}), \tag{1.14}$$

for all $[k, \boldsymbol{x}] \in G_A$. It follows from Theorem 1.2 of [6] that f_ξ is a projection in $L^1(G_A)$.

Comparing condition (iii) of a PGF$_A$ with (1.13), we see that, if $\xi \in L^2(\hat{\mathbb{R}}^n)$ is a PGF$_A$, then $w = \xi^\vee$ is a TFG$_A$ in $L^2(\mathbb{R}^n)$. Conversely, one might hope to start with a TFG$_A$ w and check whether $\hat{w}$ is a PGF$_A$. We will see, however, that the three conditions for a PGF$_A$ cannot be simultaneously achieved unless A acts on $\hat{\mathbb{R}}^n$ in a particular manner.

2. Main theorem

It is standard practice in multidimensional wavelet analysis to assume that the dilation matrix is expansive. Partly this fits the intuitive concept of dilation, but it is also an essential component of many of the proofs in the area. The theorem below illustrates the fundamental significance of this assumption.

DEFINITION 2.1. A matrix $A \in GL_n(\mathbb{R})$ is called expansive if each eigenvalue of A has absolute value greater than 1.

If either A or A^{-1} is expansive, then, for any $\boldsymbol{\gamma} \in \hat{\mathbb{R}}^n \setminus \{\mathbf{0}\}$, the A-orbit of $\boldsymbol{\gamma}$ diverges to ∞ at one end of $\mathbb{Z}$ and converges to $\mathbf{0}$ at the other end. It is exactly this property of the orbit space that permits the existence of smooth functions satisfying (iii) of 1.2. A function ϕ on $\hat{\mathbb{R}}^n$ is said to have compact support if there exists a compact set $K \subseteq \hat{\mathbb{R}}^n$ such that $\phi(\boldsymbol{\gamma}) = 0$, for all $\boldsymbol{\gamma} \in \hat{\mathbb{R}}^n \setminus K$. Let $C_c^\infty(\hat{\mathbb{R}}^n)$ denote the space of infinitely differentiable complex-valued functions on $\hat{\mathbb{R}}^n$ of compact support.

THEOREM 2.2. *Let $A \in GL_n(\mathbb{R})$. The following are equivalent:*

(a) *Either A or A^{-1} is expansive.*
(b) *There exists a TFG_A w such that $\hat{w} \in C_c^\infty(\hat{\mathbb{R}}^n)$.*
(c) *There exists a TFG_A w such that $\hat{w}$ has compact support.*
(d) *There exists a non-zero projection in $L^1(G_A)$.*
(e) *There exists a non-empty compact open subset of $\hat{G}_A$.*

PROOF. (a) $\Rightarrow$ (b) This is a modification of the proof of Theorem 2.2 of [5]. Since it is the key to understanding to construction of projections, we provide the details here. To be definite, assume that A is expansive.

Fix any norm on $\hat{\mathbb{R}}^n$ and compute the norm of matrices as operators on $\hat{\mathbb{R}}^n$ with this norm. Let $S = \{\gamma \in \hat{\mathbb{R}}^n : 1 \leq \|\gamma\| \leq \|A\|\}$. For any $\gamma \in \hat{\mathbb{R}}^n$, $\gamma \neq 0$ implies $\lim_{k \to -\infty} \|\gamma A^k\| = 0$ and $\lim_{k \to \infty} \|\gamma A^k\| = \infty$. Thus, there exists a $k_0 \in \mathbb{Z}$ such that $\|\gamma A^{k_0 - 1}\| < 1$ and $\|\gamma A^{k_0}\| \geq 1$. Then $\gamma A^{k_0} \in S$. Thus, every non-zero orbit in $\hat{\mathbb{R}}^n$ intersects the annulus S. Let $0 < \epsilon < 1$ be fixed and let

$$S_\epsilon = \{\gamma \in \hat{\mathbb{R}}^n : 1 - \epsilon \leq \|\gamma\| \leq \|A\| + \epsilon\}. \tag{2.1}$$

Let $\phi \in C_c^\infty(\hat{\mathbb{R}}^n)$ be such that $\phi(\gamma) \geq 0$, for all $\gamma \in \hat{\mathbb{R}}^n$, $\phi(\gamma) = 0$, if $\gamma \notin S_\epsilon$, and $\phi(\gamma) \geq 1$, for all $\gamma \in S$. Define

$$\sigma(\gamma) = \sum_{k=-\infty}^{\infty} |\phi(\gamma A^k)|^2, \tag{2.2}$$

for all $\gamma \in \hat{\mathbb{R}}^n \setminus \{\mathbf{0}\}$. Since $\gamma A^k \in S_\epsilon$ for only finitely many k, the sum in (2.2) is locally finite. Since $\gamma A^k \in S$ for at least one $k \in \mathbb{Z}$, $\sigma(\gamma) \geq 1$, for any $\gamma \in \hat{\mathbb{R}}^n \setminus \{\mathbf{0}\}$.

Define ξ on $\hat{\mathbb{R}}^n$ by

$$\xi(\gamma) = \begin{cases} \phi(\gamma)\sigma(\gamma)^{-1/2} & , \text{ if } \gamma \neq \mathbf{0} \\ 0 & , \text{ if } \gamma = \mathbf{0} \end{cases} \tag{2.3}$$

for all $\gamma \in \hat{\mathbb{R}}^n$. Then $\xi \in C_c^\infty(\hat{\mathbb{R}}^n)$ and there exists a $w \in L^2(\mathbb{R}^n)$ such that $\hat{w} = \xi$. By (2.2) and (2.3), $\sum_{k=-\infty}^{\infty} |\hat{w}(\gamma A^k)|^2 = 1$, for all $\gamma \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and w is a TFG_A by Proposition 1.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Let $w \in L^2(\mathbb{R}^n)$ be a TFG_A such that $\hat{w}$ can be represented by an, everywhere defined, function of compact support. Let $K \subseteq \hat{\mathbb{R}}^n$ be a compact set such that $\hat{w}(\gamma) = 0$, for all $\gamma \in \hat{\mathbb{R}}^n \setminus K$.

Suppose A has two eigenvalues λ_1 and λ_2 such that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Let V_i denote corresponding minimal invariant subspaces for A; that is, V_i is an eigenspace for A if λ_i is real, or a two-dimensional subspace on which A acts by rotation and scaling if λ_i is complex. Endow each of these subspaces with a norm such that $\|\gamma_i A^i\| = |\lambda_i| \|\gamma_i\|$, for all $\gamma_i \in V_i$, $i = 1, 2$. Further, let V_3 be the invariant subspace complementing V_1 and V_2; that is, each $\gamma \in \hat{\mathbb{R}}^n$ has a unique decomposition $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ with $\gamma_i \in V_i$. Endow V_3 with any norm. Define a particular norm $\|\cdot\|$ on $\hat{\mathbb{R}}^n$ by

$$\|\gamma\| = \max\{\|\gamma_i\| : 1 \leq i \leq 3\} \tag{2.4}$$

and let $M = \sup\{\|\gamma\| : \gamma \in K\} + 1$. Now if $\gamma \in \hat{\mathbb{R}}^n$ is any vector with $\|\gamma_1\| \geq M$ and $\|\gamma_2\| \geq M$ then for $k \in \mathbb{Z}$,

$$\|\gamma A^k\| = \max\{\, |\lambda_1|^k \|\gamma_1\|,\ |\lambda_2|^k \|\gamma_2\|,\ \|\gamma_3\|\,\} \geq M, \tag{2.5}$$

since $|\lambda_1|^k \geq 1$, if $k \geq 0$, and $|\lambda_2|^k \geq 1$, if $k < 0$. Thus, $\hat{w}(\gamma A^k) = 0$, for all $k \in \mathbb{Z}$ and all such γ, so that w could not be a TFG_A.

On the other hand, if A has an eigenvalue λ with $|\lambda| = 1$, then we choose V_1 to be a corresponding minimal invariant subspace, and V_2 its complementary invariant subspace. With norm $\|\gamma\| = \max\{\|\gamma_1\|, \|\gamma_2\|\}$ for $\gamma = \gamma_1 + \gamma_2$, $\gamma_i \in V_i$, and M as above, let γ be an arbitrary vector with $\|\gamma_1\| \geq M$. Then $\hat{w}(\gamma A^k) = 0$, for all $k \in \mathbb{Z}$ and all such γ, so that w could not be a TFG_A.

Thus, if w is a TFG_A such that $\hat{w}$ has compact support, then all the eigenvalues of A must lie on the same side of the unit circle in $\mathbb{C}$. That is, either A or A^{-1} must be expansive.

(b) $\Rightarrow$ (d) If (b) holds, then (a) also holds and our proof of (a) $\Rightarrow$ (b) shows that a TFG_A, w, may be chosen so that $\hat{w} \in C_c^\infty(\hat{\mathbb{R}}^n)$ and is supported on an annulus S_ϵ as defined in (2.1). Since A or A^{-1} is expansive, there exists a positive integer N such that if $k \in \mathbb{Z}$ and $|k| \geq N$, then

$$\hat{w}(\gamma A^k)\overline{\hat{w}}(\gamma) = 0, \text{ for all } \gamma \in \hat{\mathbb{R}}^n. \tag{2.6}$$

For any $k \in \mathbb{Z}$, $\gamma \to \hat{w}(\gamma A^k)\overline{\hat{w}}(\gamma)$ is a function in $C_c^\infty(\hat{\mathbb{R}}^n)$. Thus, there exits a $g_k \in L^1(\mathbb{R}^n)$ so that $\hat{g}_k(\gamma) = \hat{w}(\gamma A^k)\overline{\hat{w}}(\gamma)$, for all $\gamma \in \hat{\mathbb{R}}^n$. By (2.6), $g_k = 0$ if $|k| \geq N$.

Therefore, $\xi = \hat{w}$ satisfies properties (i) and (ii) of a projection generating function. Since w is a TFG_A, property (iii) is also satisfied. Thus, $\hat{w}$ is a PGF_A and $f_{\hat{w}}$, as defined in (1.14), is a non-zero projection in $L^1(G_A)$.

(d) $\Rightarrow$ (e) As remarked earlier, if f is a non-zero projection in $L^1(G_A)$, then $s(f)$ is a non-empty compact open subset of $\hat{G}_A$.

(e) $\Rightarrow$ (a) This is a minor modification of the proof of [5], 1.3 and 1.7. □

We conclude with a few remarks.

The construction in the proof of (a) $\Rightarrow$ (b) of 2.2 provides a method for generating PGF_As.

If $\mathcal{P}(\hat{\mathbb{R}}^n)$ denotes the set of all PGF_As on $\hat{\mathbb{R}}^n$ and $L^1(G_A)^P$ denotes the set of all projections in $L^1(G_A)$, the map $\xi \to f_\xi$ from $\mathcal{P}(\hat{\mathbb{R}}^n)$ into $L^1(G_A)^P$ is not one-to-one. To illustrate this, let $A = \mathrm{diag}(2, 2, \ldots, 2)$, the dilation by 2 matrix. Then the non-zero A-orbits in $\hat{\mathbb{R}}^n$ lie on rays from the origin. If $\xi \in \mathcal{P}(\hat{\mathbb{R}}^n)$ and c is any unit circle valued C^∞-function on the unit sphere in $\hat{\mathbb{R}}^n$, with respect to any smooth norm, then define

$$\eta(\gamma) = c\left(\frac{1}{\|\gamma\|}\gamma\right)\xi(\gamma), \tag{2.7}$$

for $\gamma \in \hat{\mathbb{R}}^n \setminus \{\mathbf{0}\}$ and $\eta(\mathbf{0}) = 0$. Then $f_\xi = f_\eta$.

If A is an exponential ($A = \exp(B)$ for some $n \times n$ real matrix B) and, thus, A^t is defined for all $t \in \mathbb{R}$, then one can form the semidirect product of $\mathbb{R}$ acting on $\mathbb{R}^n$, H_A say. That is, $H_A = \{[t, \boldsymbol{x}] : t \in \mathbb{R}, \boldsymbol{x} \in \hat{\mathbb{R}}^n\}$ with product

$$[t, \boldsymbol{x}][s, \boldsymbol{y}] = [t + s, A^{-s}\boldsymbol{x} + \boldsymbol{y}]. \tag{2.8}$$

Projections in $L^1(H_A)$ were studied in [5]. On the other hand, $w \in L^2(\mathbb{R}^n)$ is called a continuous tight frame generator for A (CTFG_A) if

$$\|g\|_2^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\langle g, w_{t,\boldsymbol{x}} \rangle|^2 \, d\boldsymbol{x} dt, \tag{2.9}$$

for all $g \in L^2(\mathbb{R}^n)$. Compare this with (1.10). Here

$$w_{t,\boldsymbol{x}}(\boldsymbol{z}) = \delta^{-t/2} w(A^{-t}\boldsymbol{z} - \boldsymbol{x}), \tag{2.10}$$

for all $\boldsymbol{z} \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^n$. In analogy with Theorem 2.2, one can prove that the following are equivalent:

(a) Either A or A^{-1} is expansive.
(b) There exists a CTFG_A w such that $\hat{w} \in C_c^\infty(\hat{\mathbb{R}}^n)$.
(c) There exists a CTFG_A w such that $\hat{w}$ has compact support.
(d) There exists a non-zero projection in $L^1(H_A)$.
(e) There exists a non-empty compact open subset of $\hat{H}_A$.

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