An Allegorical Semantics of Modal Logic

Kohei Kishida

Dalhousie University

20 Sept, 2018

Kripke semantics of modal logic has a successful model theory:
e.g. bisimulation theorems, correspondence theory, duality theory.

Goals

- Give structural accounts of the model theory.
  —**Rel** will do the job.

Kripke semantics of modal logic has a successful model theory:
e.g. bisimulation theorems, correspondence theory, duality theory.

Goals

- Give structural accounts of the model theory.

  —**Rel** will do the job.
- **Rel** has many generalizations. Identify which accommodates the model theory.

  —Allegories, i.e. the categories of relations of regular categories.
- In effect, Kripke semantics will be extended to regular categories.

Kripke semantics of modal logic has a successful model theory:
e.g. bisimulation theorems, correspondence theory, duality theory.

Goals

- Give structural accounts of the model theory.
  —**Rel** will do the job.
- **Rel** has many generalizations. Identify which accommodates the model theory.
  —Allegories, i.e. the categories of relations of regular categories.
- In effect, Kripke semantics will be extended to regular categories.

Outline

1. Recast Kripke semantics and its model theory using **Rel**.
2. Briefly review allegories.
3. Give allegorical semantics of modal logic, and model theory.

Kripke Semantics

Interprets propositional logic + modal operators $\Box_i$, $\Diamond_i$ ($i \in I$).

Kripke Semantics

Interprets propositional logic + modal operators $\Box_i$, $\Diamond_i$ ($i \in I$).

Two layers of semantic structures:

- A Kripke frame, a set X plus $R_i : X \nrightarrow X$.
  Each R_i interprets $\Box_i$, $\Diamond_i$.
- A Kripke model, a frame (X, R_i) plus $[\![p]\!] \subseteq X$.
  Each $[\![p]\!]$ interprets a prop. variable p.

Kripke Semantics

Interprets propositional logic + modal operators $\Box_i$, $\Diamond_i$ $(i \in I)$.

Two layers of semantic structures:

- A Kripke frame, a set X plus $R_i : X \nrightarrow X$.
  Each R_i interprets $\Box_i$, $\Diamond_i$.
- A Kripke model, a frame (X, R_i) plus $[\![p]\!] \subseteq X$.
  Each $[\![p]\!]$ interprets a prop. variable p.

$x \vDash \varphi$ "φ is true at x", for a world / state $x \in X$ and a formula φ.

Kripke Semantics

Interprets propositional logic + modal operators $\Box_i$, $\Diamond_i$ $(i \in I)$.

Two layers of semantic structures:

- A Kripke frame, a set X plus $R_i : X \nrightarrow X$.
  Each R_i interprets $\Box_i$, $\Diamond_i$.
- A Kripke model, a frame (X, R_i) plus $[\![p]\!] \subseteq X$.
  Each $[\![p]\!]$ interprets a prop. variable p.

$x \vDash \varphi$ "φ is true at x", for a world / state $x \in X$ and a formula φ.

$$
\begin{aligned}
x \vDash p &\iff x \in [\![p]\!] && \text{(via the model)},\\
x \vDash \varphi \wedge \psi &\iff x \vDash \varphi \text{ and } x \vDash \psi, &&\\
x \vDash \Box_i\varphi &\iff y \vDash \varphi \text{ for all } y \text{ s.th. } xR_iy && \text{(via the frame)},\\
x \vDash \Diamond_i\varphi &\iff y \vDash \varphi \text{ for some } y \text{ s.th. } xR_iy && \text{(via the frame)}.
\end{aligned}
$$

"Standard translation": "$x \models \varphi$" $\xrightarrow{\text{tr}}$ $\varphi(x)$

$$\begin{aligned}
\operatorname{tr}(p) &= Px, \\
\operatorname{tr}(\varphi \wedge \psi) &= \operatorname{tr}(\varphi) \wedge \operatorname{tr}(\psi), \\
\operatorname{tr}(\Box_i \varphi) &= \forall y.\, R_i xy \Rightarrow \operatorname{tr}(\varphi)[y/x], \\
\operatorname{tr}(\Diamond_i \varphi) &= \exists y.\, R_i xy \wedge \operatorname{tr}(\varphi)[y/x].
\end{aligned}$$

"Standard translation": "$x \models \varphi$" $\xrightsquigarrow{\mathrm{tr}}$ $\varphi(x)$

$$\begin{aligned}
\mathrm{tr}(p) &= Px, \\
\mathrm{tr}(\varphi \wedge \psi) &= \mathrm{tr}(\varphi) \wedge \mathrm{tr}(\psi), \\
\mathrm{tr}(\Box_i \varphi) &= \forall y.\, R_i xy \Rightarrow \mathrm{tr}(\varphi)[y/x], \\
\mathrm{tr}(\Diamond_i \varphi) &= \exists y.\, R_i xy \wedge \mathrm{tr}(\varphi)[y/x].
\end{aligned}$$

Two layers of semantic structures $\Longrightarrow$ two (split) perspectives:

- **Bisimulation theorems**:

  "modal logic is about LTSs (Kripke models)."
- **Correspondence theory**:

  "modal logic is about binary relations (Kripke frames)."

"Standard translation": "$x \vDash \varphi$" $\overset{\mathrm{tr}}{\rightsquigarrow}$ $\varphi(x)$

$$
\begin{aligned}
\mathrm{tr}(p) &= Px, \\
\mathrm{tr}(\varphi \wedge \psi) &= \mathrm{tr}(\varphi) \wedge \mathrm{tr}(\psi), \\
\mathrm{tr}(\Box_i \varphi) &= \forall y.\, R_i x y \Rightarrow \mathrm{tr}(\varphi)[y/x], \\
\mathrm{tr}(\Diamond_i \varphi) &= \exists y.\, R_i x y \wedge \mathrm{tr}(\varphi)[y/x].
\end{aligned}
$$

Two layers of semantic structures $\Longrightarrow$ two (split) perspectives:

- **Bisimulation theorems**:
  "modal logic is about LTSs (Kripke models)."
- **Correspondence theory**:
  "modal logic is about binary relations (Kripke frames)."

Also, • **Duality theory**:
Kripke frames $\simeq$ (powerset algebras with operators)$^{\mathrm{op}}$.

"Standard translation": "$x \vDash \varphi$" $\overset{\mathrm{tr}}{\rightsquigarrow} \varphi(x)$

$$\begin{aligned} \mathrm{tr}(p) &= Px, \\ \mathrm{tr}(\varphi \wedge \psi) &= \mathrm{tr}(\varphi) \wedge \mathrm{tr}(\psi), \\ \mathrm{tr}(\Box_i \varphi) &= \forall y.\, R_i xy \Rightarrow \mathrm{tr}(\varphi)[y/x], \\ \mathrm{tr}(\Diamond_i \varphi) &= \exists y.\, R_i xy \wedge \mathrm{tr}(\varphi)[y/x]. \end{aligned}$$

Two layers of semantic structures $\Longrightarrow$ two (split) perspectives:

- **Bisimulation theorems**:
  "modal logic is about LTSs (Kripke models)."
- **Correspondence theory**:
  "modal logic is about binary relations (Kripke frames)."

Also, • **Duality theory**:

Kripke frames $\simeq$ (powerset algebras with operators)$^{\mathrm{op}}$.

Rel gives a more unifying approach to these perspectives.

Also, **some variants of modal logic**:

- Temporal logic has modalities about the future and about the past, i.e. modalities of opposite relations.
- Dynamic logic has composition and union of transitions.
- "Dynamic epistemic logic" has modalities of transitions across different models.
- Different $\vdash_\sigma$ for different stages σ of computation (e.g. quote and unquote as modalities).

Thus we need involution, union, etc., and categorification—hence **Rel**.

Semantics Using Rel (take 1)

Every relation $R : X \nrightarrow Y$ induces two adjoint pairs:

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left=2, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left=2, "\forall_{R^\dagger}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[r, shift right=2, "\forall_R"'] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift right=2, "\exists_{R^\dagger}"'] \end{tikzcd}$$

$$\exists_R(S) = \{\, v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv \,\},$$
$$\forall_R(S) = \{\, v \in Y \mid w \in S \text{ for all } w \text{ s.th. } wRv \,\}.$$

Semantics Using Rel (take 1)

Every relation $R : X \nrightarrow Y$ induces two adjoint pairs:

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left=1.5ex, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left=1.5ex, "\forall_{R^\dagger}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[r, shift right=1.5ex, "\forall_R"'] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift right=1.5ex, "\exists_{R^\dagger}"'] \end{tikzcd}$$

$$\exists_R(S) = \{\, v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv \,\},$$
$$\forall_R(S) = \{\, v \in Y \mid w \in S \text{ for all } w \text{ s.th. } wRv \,\}.$$

E.g. For $R = f$ a function, $\exists_f \dashv \forall_{f^\dagger} = f^{-1} = \exists_{f^\dagger} \dashv \forall_f$.

Semantics Using Rel (take 1)

Every relation $R : X \nrightarrow Y$ induces two adjoint pairs:

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left=1.5ex, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left=1.5ex, "\forall_{R^\dagger}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[r, shift right=1.5ex, "\forall_R"'] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift right=1.5ex, "\exists_{R^\dagger}"'] \end{tikzcd}$$

$$\exists_R(S) = \{ v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv \},$$
$$\forall_R(S) = \{ v \in Y \mid w \in S \text{ for all } w \text{ s.th. } wRv \}.$$

E.g. For $R = f$ a function, $\exists_f \dashv \forall_{f^\dagger} = f^{-1} = \exists_{f^\dagger} \dashv \forall_f$.

E.g. $[\![\Diamond\varphi]\!] = \exists_{R^\dagger}[\![\varphi]\!]$ and $[\![\Box\varphi]\!] = \forall_{R^\dagger}[\![\varphi]\!]$ for $R : X \nrightarrow X$.
We write $\blacklozenge$ and $\blacksquare$ for the opposite, $\exists_R$ and $\forall_R$.

Semantics Using Rel (take 1)

Every relation $R : X \nrightarrow Y$ induces two adjoint pairs:

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left=2, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left=2, "\forall_{R^\dagger}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[r, shift right=2, "\forall_R"'] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift right=2, "\exists_{R^\dagger}"'] \end{tikzcd}$$

$$\exists_R(S) = \{\, v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv \,\},$$
$$\forall_R(S) = \{\, v \in Y \mid w \in S \text{ for all } w \text{ s.th. } wRv \,\}.$$

E.g. For $R = f$ a function, $\exists_f \dashv \forall_{f^\dagger} = f^{-1} = \exists_{f^\dagger} \dashv \forall_f$.

E.g. $[\![\Diamond\varphi]\!] = \exists_{R^\dagger}[\![\varphi]\!]$ and $[\![\Box\varphi]\!] = \forall_{R^\dagger}[\![\varphi]\!]$ for $R : X \nrightarrow X$.
We write $\blacklozenge$ and $\blacksquare$ for the opposite, $\exists_R$ and $\forall_R$.

Complete atomic Boolean algebras ("caBas", $\simeq$ powerset algebras):

- **caBa**$_\vee$ with all-$\vee$-preserving maps,
- **caBa**$_\wedge$ with all-$\wedge$-preserving maps.

Then $\exists_- : \mathbf{Rel} \to \mathbf{caBa}_\vee$ and $\forall_- : \mathbf{Rel} \to \mathbf{caBa}_\wedge$, and moreover . . . .

$\exists_{-} : \mathbf{Rel} \to \mathbf{caBa}_{\vee}$ and $\forall_{-} : \mathbf{Rel} \to \mathbf{caBa}_{\wedge}$ are (1-) equivalences.

$\exists_{\_} : \mathbf{Rel} \to \mathbf{caBa}_{\vee}$ and $\forall_{\_} : \mathbf{Rel} \to \mathbf{caBa}_{\wedge}$ are (1-) equivalences.

Thm (Thomason 1975)**.**

$$\text{Kripke frames} \simeq (\text{caBas with } \vee\text{-preserving operators})^{\mathrm{op}}.$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[u, "R", "\shortmid"{marking}] \arrow[r, "f"'] \arrow[ur, phantom, "\leq"] & Y \arrow[u, "S"', "\shortmid"{marking}] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[d, "\exists_{R^\dagger}"'] & \mathcal{P}Y \arrow[l, "f^{-1}"'] \arrow[d, "\exists_{S^\dagger}"] \arrow[dl, phantom, "\leq"] \\ \mathcal{P}X & \mathcal{P}Y \arrow[l, "f^{-1}"] \end{tikzcd}$$

$\exists_{\_} : \mathbf{Rel} \to \mathbf{caBa}_{\vee}$ and $\forall_{\_} : \mathbf{Rel} \to \mathbf{caBa}_{\wedge}$ are (1-) equivalences.

Thm (Thomason 1975)**.**

Kripke frames $\simeq$ (caBas with $\vee$-preserving operators)$^{\mathrm{op}}$.

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X \arrow[u, "R", "|" marking] \arrow[r, "f"'] \arrow[ur, phantom, "\leq"] & Y \arrow[u, "S"', "|" marking]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}X \arrow[d, "\exists_{R^\dagger}"'] & \mathcal{P}Y \arrow[l, "f^{-1}"'] \arrow[d, "\exists_{S^\dagger}"] \arrow[dl, phantom, "\geq"] \\
\mathcal{P}X & \mathcal{P}Y \arrow[l, "f^{-1}"]
\end{tikzcd}
$$

Thm. Bisimulations preserve satisfaction.

Pf. Because they are spans of homomorphisms.

$$
\begin{tikzcd}
X & Z \arrow[l, "f"'] \arrow[r, "g"] & Y \\
X \arrow[u, "R", "|" marking] & Z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, "U"', "|" marking] \arrow[ul, phantom, "\geq"] \arrow[ur, phantom, "\leq"] & Y \arrow[u, "S"', "|" marking]
\end{tikzcd}
$$

Rel is moreover enriched in **Pos**.

$\mathbf{Rel}$ is moreover enriched in $\mathbf{Pos}$.

- $\exists_{-} : \mathbf{Rel} \longrightarrow \mathbf{caBa}_{\vee}$ is a 2-equivalence.
- $\exists_{-^{\dagger}} : \mathbf{Rel}^{\mathrm{op}} \longrightarrow \mathbf{caBa}_{\vee}$ is a 1-cell duality.
- $\forall_{-} : \mathbf{Rel}^{\mathrm{co}} \longrightarrow \mathbf{caBa}_{\wedge}$ is a 2-cell duality.
- $\forall_{-^{\dagger}} : \mathbf{Rel}^{\mathrm{coop}} \longrightarrow \mathbf{caBa}_{\wedge}$ is a biduality.

Rel is moreover enriched in **Pos**.

- $\exists_{\_} : \mathbf{Rel} \to \mathbf{caBa}_{\vee}$ is a 2-equivalence.
- $\exists_{\_^{\dagger}} : \mathbf{Rel}^{\mathrm{op}} \to \mathbf{caBa}_{\vee}$ is a 1-cell duality.
- $\forall_{\_} : \mathbf{Rel}^{\mathrm{co}} \to \mathbf{caBa}_{\wedge}$ is a 2-cell duality.
- $\forall_{\_^{\dagger}} : \mathbf{Rel}^{\mathrm{coop}} \to \mathbf{caBa}_{\wedge}$ is a biduality.

Thm (Lemmon-Scott 1977). $(R^n)^{\dagger};R^m \subseteq R^{\ell};(R^k)^{\dagger}$ corresponds to

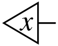
$$\Diamond^m \Box^k \varphi \vdash \Box^n \Diamond^{\ell} \varphi, \qquad \Diamond^n \Box^{\ell} \varphi \vdash \Box^m \Diamond^k \varphi.$$

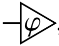
Pf.

$$\begin{array}{c}
(R^n)^{\dagger};R^m \subseteq R^{\ell};(R^k)^{\dagger} \\ \hline\hline
\blacklozenge^n \circ \Diamond^m \leqslant \Diamond^{\ell} \circ \blacklozenge^k \\ \hline\hline
\Diamond^m \leqslant \Box^n \circ \Diamond^{\ell} \circ \blacklozenge^k \\ \hline\hline
\Diamond^m \circ \Box^k \leqslant \Box^n \circ \Diamond^{\ell}
\end{array}
\qquad
\begin{array}{c}
(R^n)^{\dagger};R^m \subseteq R^{\ell};(R^k)^{\dagger} \\ \hline\hline
\Box^{\ell} \circ \blacksquare^k \leqslant \blacksquare^n \circ \Box^m \\ \hline\hline
\Diamond^n \circ \Box^{\ell} \circ \blacksquare^k \leqslant \Box^m \\ \hline\hline
\Diamond^n \circ \Box^{\ell} \leqslant \Box^m \circ \Diamond^k
\end{array}$$

E.g.
- $\varphi \vdash \Diamond\varphi$, $\Box\varphi \vdash \varphi \iff 1 \subseteq R$ (reflexivity);
- $\Diamond\Diamond\varphi \vdash \Diamond\varphi$, $\Box\varphi \vdash \Box\Box\varphi \iff R;R \subseteq R$ (transitivity);
- $\varphi \vdash \Box\Diamond\varphi$, $\Diamond\Box\varphi \vdash \varphi \iff R^{\dagger} \subseteq R$ (symmetry).

Semantics in Rel (take 2)

Worlds $x \in X$ are functions $x : 1 \to X$, or , "states".

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or , "effects".

Semantics in Rel (take 2)

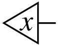
Worlds $x \in X$ are functions $x : 1 \to X$, or $\lhd x -$, "states".

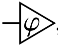
Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or $- \varphi \rhd$, "effects".

So the three components of Kripke frames and models become

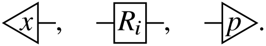
$$\lhd x -, \quad -\boxed{R_i}-, \quad - p \rhd.$$

Semantics in Rel (take 2)

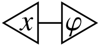
Worlds $x \in X$ are functions $x : 1 \to X$, or , "states".

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or , "effects".

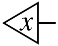
So the three components of Kripke frames and models become

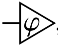


The truth of $x \vDash \varphi$ is given by the "generalized Born rule":

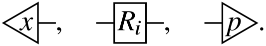


Semantics in Rel (take 2)

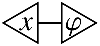
Worlds $x \in X$ are functions $x : 1 \to X$, or , "states".

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or , "effects".

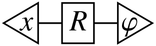
So the three components of Kripke frames and models become



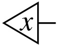
The truth of $x \vDash \varphi$ is given by the "generalized Born rule":

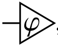


$x \vDash \Diamond\varphi$ is

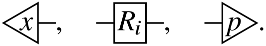


Semantics in Rel (take 2)

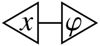
Worlds $x \in X$ are functions $x : 1 \to X$, or , "states".

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or , "effects".

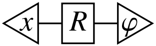
So the three components of Kripke frames and models become



The truth of $x \vDash \varphi$ is given by the "generalized Born rule":

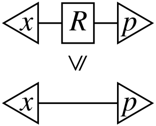


$x \vDash \Diamond\varphi$ is

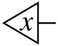


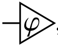
Validity of $p \vdash \Diamond p$ in a Kripke frame is

$\forall x, p$

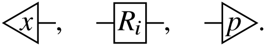


Semantics in Rel (take 2)

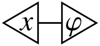
Worlds $x \in X$ are functions $x : 1 \to X$, or , "states".

Propositions $\varphi \subseteq X$ are relations $\varphi : X \nrightarrow 1$, or , "effects".

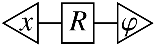
So the three components of Kripke frames and models become



The truth of $x \vDash \varphi$ is given by the "generalized Born rule":

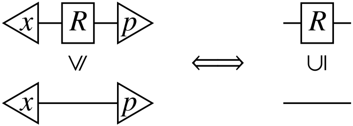


$x \vDash \Diamond\varphi$ is



Validity of $p \vdash \Diamond p$ in a Kripke frame is

$\forall x, p$



Allegories

There are many categorical generalizations of **Rel**. Which of them admits the foregoing approach to modal logic? — Allegories!

Def. An allegory $\mathcal{A}$ is a **Pos**-enriched †-category in which

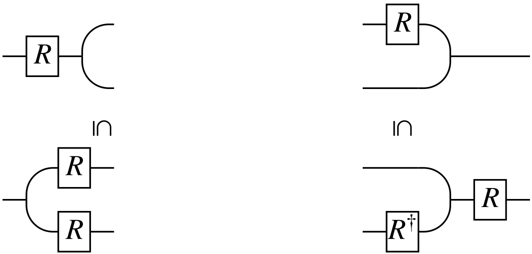
- each $\mathcal{A}(X,Y)$ has a binary meet,
- † preserves $\subseteq$ and $\cap$,
- semi-distributivity: $R;(S \cap T) \subseteq (R;S) \cap (R;T)$,
- the modular law: $(S;R) \cap T \subseteq (S \cap (T;R^\dagger));R$.

Allegories

There are many categorical generalizations of **Rel**. Which of them admits the foregoing approach to modal logic? — Allegories!

Def. An allegory $\mathcal{A}$ is a **Pos**-enriched †-category in which

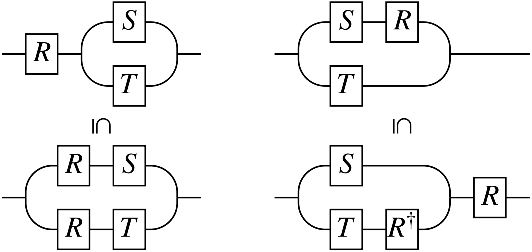
- each $\mathcal{A}(X,Y)$ has a binary meet,
- † preserves $\subseteq$ and $\cap$,
- semi-distributivity: $R;(S \cap T) \subseteq (R;S) \cap (R;T)$,
- the modular law: $(S;R) \cap T \subseteq (S \cap (T;R^\dagger));R$.



There are many categorical generalizations of **Rel**. Which of them admits the foregoing approach to modal logic? — Allegories!

Def. An allegory $\mathcal{A}$ is a **Pos**-enriched †-category in which

- each $\mathcal{A}(X,Y)$ has a binary meet,
- † preserves $\subseteq$ and $\cap$,
- semi-distributivity: $R;(S \cap T) \subseteq (R;S) \cap (R;T)$,
- the modular law: $(S;R) \cap T \subseteq (S \cap (T;R^{\dagger}));R$.



$R : X \nrightarrow X$ is
- reflexive if $1_X \subseteq R$,
- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

$R : X \nrightarrow Y$ is
- total if $1_X \subseteq R;R^\dagger$,
- simple, or is a partial map, if $R^\dagger;R \subseteq 1_Y$,
- a map if it is total and simple (i.e. if it is a left adjoint).

$R : X \nrightarrow X$ is
- reflexive if $1_X \subseteq R$,
- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

$R : X \nrightarrow Y$ is
- total if $1_X \subseteq R;R^\dagger$,
- simple, or is a partial map, if $R^\dagger;R \subseteq 1_Y$,
- a map if it is total and simple (i.e. if it is a left adjoint).

$$\mathcal{A} \longmapsto \mathbf{Map}(\mathcal{A})$$
$$\mathbf{Rel}(\mathcal{C}) \longleftarrow\!\!| \; \mathcal{C}$$

$R : X \nrightarrow X$ is

- reflexive if $1_X \subseteq R$,
- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

$R : X \nrightarrow Y$ is

- total if $1_X \subseteq R;R^\dagger$,
- simple, or is a partial map, if $R^\dagger;R \subseteq 1_Y$,
- a map if it is total and simple (i.e. if it is a left adjoint).

Fact.

$$\begin{array}{ccc} \mathcal{A} & \longmapsto & \mathbf{Map}(\mathcal{A}) \\ & \cong & \\ \mathbf{Rel}(\mathcal{C}) & \longleftarrow\!\!\!\shortmid & \mathcal{C} \end{array}$$

allegories	categories
unital and tabular	regular

Def. $\mathcal{A}$ is unital if it has a "unit" ($\approx$ a terminal obj. of $\mathbf{Map}(\mathcal{A})$).

Def. $\mathcal{A}$ is tabular if every relation is "tabulated" by a jointly monic pair of maps.

$R : X \nrightarrow X$ is • reflexive if $1_X \subseteq R$,

- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

$R : X \nrightarrow Y$ is • total if $1_X \subseteq R;R^\dagger$,

- simple, or is a partial map, if $R^\dagger;R \subseteq 1_Y$,
- a map if it is total and simple (i.e. if it is a left adjoint).

Fact.

$$\begin{array}{ccc} \mathcal{A} & \longmapsto & \mathbf{Map}(\mathcal{A}) \\ & \cong & \\ \mathbf{Rel}(\mathcal{C}) & \longleftarrow\!\!\dashv & \mathcal{C} \end{array}$$

allegories	categories	logic
unital and tabular	regular	$\top, \wedge, \exists, =$

Def. $\mathcal{A}$ is unital if it has a "unit" ($\approx$ a terminal obj. of $\mathbf{Map}(\mathcal{A})$).

Def. $\mathcal{A}$ is tabular if every relation is "tabulated" by a jointly monic pair of maps.

$R : X \nrightarrow X$ is • reflexive if $1_X \subseteq R$,

- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

$R : X \nrightarrow Y$ is • total if $1_X \subseteq R;R^\dagger$,

- simple, or is a partial map, if $R^\dagger;R \subseteq 1_Y$,
- a map if it is total and simple (i.e. if it is a left adjoint).

Fact.

$$\begin{array}{ccc} \mathcal{A} & \longmapsto & \mathbf{Map}(\mathcal{A}) \\ & \cong & \\ \mathbf{Rel}(\mathcal{C}) & \longleftarrow\!\shortmid & \mathcal{C} \end{array}$$

allegories	categories	logic
unital and tabular	regular	$\top, \wedge, \exists, =$
+ "distributive"	coherent (pre-logoi)	$\bot, \vee$
+ "division"	Heyting (logoi)	$\Rightarrow, \forall$
+ "power"	topoi	$\in$

Def. $\mathcal{A}$ is unital if it has a "unit" ($\approx$ a terminal obj. of $\mathbf{Map}(\mathcal{A})$).

Def. $\mathcal{A}$ is tabular if every relation is "tabulated" by a jointly monic pair of maps.

$R : X \nrightarrow X$ is • reflexive if $1_X \subseteq R$,

- transitive if $R;R \subseteq R$,
- symmetric if $R^\dagger \subseteq R$.

$R : X \nrightarrow Y$ is • total if $1_X \subseteq R;R^\dagger$,

- simple, or is a partial map, if $R^\dagger;R \subseteq 1_Y$,
- a map if it is total and simple (i.e. if it is a left adjoint).

Fact.

$$\begin{tikzcd} \mathcal{A} \arrow[r, mapsto] & \mathbf{Map}(\mathcal{A}) \\ \mathbf{Rel}(\mathcal{C}) \arrow[u, phantom, ""] & \mathcal{C} \arrow[l, mapsto] \arrow[u, phantom, ""] \end{tikzcd} \qquad \cong$$

allegories	categories	logic
unital and tabular	regular	$\top, \wedge, \exists, =, \Diamond$
+ "distributive"	coherent (pre-logoi)	$\bot, \vee$
+ "division"	Heyting (logoi)	$\Rightarrow, \forall, \Box$
+ "power"	topoi	$\in$

Def. $\mathcal{A}$ is unital if it has a "unit" ($\approx$ a terminal obj. of $\mathbf{Map}(\mathcal{A})$).

Def. $\mathcal{A}$ is tabular if every relation is "tabulated" by a jointly monic pair of maps.

Subobjects

Two allegorical expressions for $\mathrm{Sub}_{\mathbf{Map}(\mathcal{A})}(X)$:

- $R : X \nrightarrow X$ is correflexive, or is a "core", if $R \subseteq 1_X$.
  $\mathrm{Cor}(X)$, the cores on X.
- $\mathcal{A}(X, 1)$, the effects on X.

Subobjects

Two allegorical expressions for $\mathrm{Sub}_{\mathbf{Map}(\mathcal{A})}(X)$:

- $R : X \nrightarrow X$ is correflexive, or is a "core", if $R \subseteq 1_X$.
  $\mathrm{Cor}(X)$, the cores on X.
- $\mathcal{A}(X, 1)$, the effects on X.

Fact. In a unital allegory $\mathcal{A}$, define

$$
\begin{tikzcd}
S \arrow[d, mapsto] \\
\overline{S} = S;S^\dagger \cap 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{A}(X,Y) \arrow[dl, "\overline{\cdot}"] \arrow[dr, "\widehat{\cdot}"] & \\
\mathrm{Cor}(X) \arrow[rr, shift left, "\widehat{\cdot}"] & & \mathcal{A}(X,1) \arrow[ll, shift left, "\overline{\cdot}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, mapsto] \\
\widehat{S} = S;\top_{(Y,1)}
\end{tikzcd}
$$

($\top_{(Y,1)}$ is the top element of $\mathcal{A}(Y, 1)$, which exists in a unital $\mathcal{A}$.)

Subobjects

Two allegorical expressions for $\mathrm{Sub}_{\mathbf{Map}(\mathcal{A})}(X)$:

- $R : X \nrightarrow X$ is correflexive, or is a "core", if $R \subseteq 1_X$.
  $\mathrm{Cor}(X)$, the cores on X.
- $\mathcal{A}(X, 1)$, the effects on X.

Fact. In a unital allegory $\mathcal{A}$, define

$$
\begin{tikzcd}
S \arrow[d, mapsto] \\
\overline{S} = S;S^{\dagger} \cap 1
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathcal{A}(X,Y) \arrow[dl, "\overline{\cdot}"'] \arrow[dr, "\widehat{\cdot}"] & \\
\mathrm{Cor}(X) \arrow[rr, shift left, "\widehat{\cdot}"] & & \mathcal{A}(X,1) \arrow[ll, shift left, "\overline{\cdot}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, mapsto] \\
\widehat{S} = S;\top_{(Y,1)}
\end{tikzcd}
$$

($\top_{(Y,1)}$ is the top element of $\mathcal{A}(Y, 1)$, which exists in a unital $\mathcal{A}$.)
Then the diagram commutes; the bottom edges are isomorphisms.
If moreover $\mathcal{A}$ is tabular, $\mathrm{Cor}(X) \cong \mathcal{A}(X, 1) \cong \mathrm{Sub}_{\mathbf{Map}(\mathcal{A})}(X)$.

Def. $\mathcal{A}$ is distributive if each $\mathcal{A}(X,Y)$ is a distributive lattice and pre- and post-compositions preserve $\cup$.

Def. $\mathcal{A}$ is distributive if each $\mathcal{A}(X,Y)$ is a distributive lattice and pre- and post-compositions preserve $\cup$.

Def. $\mathcal{A}$ is a division allegory if compositions have right adjoints. For $R : X \nrightarrow Y$,

$$\mathcal{A}(Y,Z) \underset{R\backslash -}{\overset{R;-}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}(X,Z) \qquad \mathcal{A}(Z,X) \underset{-/R}{\overset{-;R}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}(Z,Y)$$

$$\frac{R;S \subseteq T}{\overline{S \subseteq R\backslash T}} \qquad \frac{S;R \subseteq T}{\overline{S \subseteq T/R}}$$

Def. $\mathcal{A}$ is distributive if each $\mathcal{A}(X,Y)$ is a distributive lattice and pre- and post-compositions preserve $\cup$.

Def. $\mathcal{A}$ is a division allegory if compositions have right adjoints. For $R : X \nrightarrow Y$,

$$\begin{tikzcd} \mathcal{A}(Y,Z) \arrow[r, shift left, "R;-"] \arrow[r, phantom, "\perp"] & \mathcal{A}(X,Z) \arrow[l, shift left, "R\backslash -"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(Z,X) \arrow[r, shift left, "-;R"] \arrow[r, phantom, "\perp"] & \mathcal{A}(Z,Y) \arrow[l, shift left, "-/R"] \end{tikzcd}$$

$$\frac{R;S \subseteq T}{S \subseteq R\backslash T} \qquad \frac{S;R \subseteq T}{S \subseteq T/R}$$

E.g.

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, shift left, "\exists_{R^\dagger} = R;-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, shift left, "\forall_R = R\backslash -"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, shift left, "\exists_R = R^\dagger;-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y) \arrow[l, shift left, "\forall_{R^\dagger} = R^\dagger\backslash - = (-^\dagger/R)^\dagger"] \end{tikzcd}$$

Def. $\mathcal{A}$ is distributive if each $\mathcal{A}(X,Y)$ is a distributive lattice and pre- and post-compositions preserve $\cup$.

Def. $\mathcal{A}$ is a division allegory if compositions have right adjoints. For $R : X \nrightarrow Y$,

$$\mathcal{A}(Y,Z) \underset{R\backslash -}{\overset{R;-}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}(X,Z) \qquad \mathcal{A}(Z,X) \underset{-/R}{\overset{-;R}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}(Z,Y)$$

$$\frac{R;S \subseteq T}{S \subseteq R\backslash T} \qquad \frac{S;R \subseteq T}{S \subseteq T/R}$$

E.g.

$$\mathcal{P}(Y) \underset{\forall_R = R\backslash -}{\overset{\exists_{R^\dagger} = R;-}{\underset{\perp}{\rightleftarrows}}} \mathcal{P}(X) \qquad \mathcal{P}(X) \underset{\forall_{R^\dagger} = R^\dagger\backslash - = (-^\dagger/R)^\dagger}{\overset{\exists_R = R^\dagger;-}{\underset{\perp}{\rightleftarrows}}} \mathcal{P}(Y)$$

We extend this and write

$$\mathcal{A}(Y,1) \underset{\forall_R = R\backslash -}{\overset{\exists_{R^\dagger} = R;-}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}(X,1) \qquad \mathcal{A}(X,1) \underset{\forall_{R^\dagger} = R^\dagger\backslash -}{\overset{\exists_R = R^\dagger;-}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}(Y,1)$$

Allegorical Semantics

The interpretation on the cores $\mathrm{Cor}(X)$ amounts to the following on the effects $\mathcal{A}(X,1)$:

$$
\begin{aligned}
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] = \overline{[\![\varphi]\!]};[\![\psi]\!], \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!], \\
[\![\varphi \Rightarrow \psi]\!] &= \overline{[\![\varphi]\!]} \backslash [\![\psi]\!], \\
[\![\neg\varphi]\!] &= [\![\varphi \Rightarrow \bot]\!], \\
[\![\top]\!] &= \top_{(X,1)}, \\
[\![\bot]\!] &= \bot_{(X,1)}.
\end{aligned}
$$

The interpretation on the cores $\mathrm{Cor}(X)$ amounts to the following on the effects $\mathcal{A}(X, 1)$:

$$
\begin{aligned}
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] = \overline{[\![\varphi]\!]};[\![\psi]\!], \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!], \\
[\![\varphi \Rightarrow \psi]\!] &= \overline{[\![\varphi]\!]} \backslash [\![\psi]\!], \\
[\![\neg\varphi]\!] &= [\![\varphi \Rightarrow \bot]\!], \\
[\![\top]\!] &= \top_{(X,1)}, \\
[\![\bot]\!] &= \bot_{(X,1)}.
\end{aligned}
$$

To this, add, for each $R_i : X \nrightarrow X$,

$$
\begin{aligned}
[\![\Diamond_i \varphi]\!] &= R_i;[\![\varphi]\!], \\
[\![\Box_i \varphi]\!] &= {R_i}^{\dagger} \backslash [\![\varphi]\!].
\end{aligned}
$$

Syntax

- Basic types τ.
- Each prop. variable p has a basic type $p : \tau$.
- Each label i of modal operators has a type $i : \tau \to \tau'$.
- Different prop. constants $\top_\tau, \bot_\tau : \tau$ for each different τ.

Syntax

- Basic types τ.
- Each prop. variable p has a basic type $p : \tau$.
- Each label i of modal operators has a type $i : \tau \to \tau'$.
- Different prop. constants $\top_\tau, \bot_\tau : \tau$ for each different τ.

$$\frac{}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash p, \top_\tau, \bot_\tau : \tau} \qquad \frac{}{\vdash i : \tau \to \tau'}$$

Syntax

- Basic types τ.
- Each prop. variable p has a basic type $p : \tau$.
- Each label i of modal operators has a type $i : \tau \to \tau'$.
- Different prop. constants $\top_\tau, \bot_\tau : \tau$ for each different τ.

$$\frac{}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash p, \top_\tau, \bot_\tau : \tau} \qquad \frac{}{\vdash i : \tau \to \tau'}$$

$$\frac{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi : \tau \qquad p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \psi : \tau}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi \wedge \psi, \varphi \vee \psi, \varphi \Rightarrow \psi : \tau}$$

$$\frac{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi : \tau}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \neg\varphi : \tau}$$

Syntax

- Basic types τ.
- Each prop. variable p has a basic type $p : \tau$.
- Each label i of modal operators has a type $i : \tau \to \tau'$.
- Different prop. constants $\top_\tau, \bot_\tau : \tau$ for each different τ.

$$\frac{}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash p, \top_\tau, \bot_\tau : \tau} \qquad \frac{}{\vdash i : \tau \to \tau'}$$

$$\frac{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi : \tau \qquad p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \psi : \tau}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi \wedge \psi, \varphi \vee \psi, \varphi \Rightarrow \psi : \tau}$$

$$\frac{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi : \tau}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \neg\varphi : \tau}$$

$$\frac{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \varphi : \tau \qquad \vdash i : \tau \to \tau'}{p_1 : \tau_1, \ldots, p_n : \tau_n \vdash \Diamond_i\varphi, \Box_i\varphi : \tau'}$$

Frames and Models

Generate a category **D** from basic types τ and labels $i : \tau \to \tau'$.

Frames and Models

Generate a category $\mathbf{D}$ from basic types τ and labels $i : \tau \to \tau'$.

Kripke frames can then be generalized by

Def. A frame diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}^{\mathrm{op}} \to \mathcal{A}$.

$$
\begin{array}{cccc}
\tau & [\![\tau]\!] & \mathcal{A}([\![\tau]\!], 1) & [\![\varphi]\!] \\
i\big\downarrow & [\![i]\!]\big\uparrow & \big\downarrow [\![i]\!];- & \big\downarrow \\
\tau' & [\![\tau']\!] & \mathcal{A}([\![\tau']\!], 1) & [\![\Diamond_i \varphi]\!]
\end{array}
$$

Frames and Models

Generate a category $\mathbf{D}$ from basic types τ and labels $i : \tau \to \tau'$.

Kripke frames can then be generalized by

Def. A frame diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}^{\mathrm{op}} \to \mathcal{A}$.

$$
\begin{tikzcd}
\tau \arrow[d, "i"'] \\
\tau'
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\![\tau]\!]} \\
{[\![\tau']\!]} \arrow[u, "{[\![i]\!]}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
{\mathcal{A}([\![\tau]\!], 1)} \arrow[d, "{[\![i]\!];-}"] \\
{\mathcal{A}([\![\tau']\!], 1)}
\end{tikzcd}
\quad
\begin{tikzcd}
{[\![\varphi]\!]} \arrow[d, mapsto] \\
{[\![\Diamond_i \varphi]\!]}
\end{tikzcd}
$$

Let $\mathbf{D}_*$ be $\mathbf{D}$ with an object $*$ and labels $p : * \to \tau$ added.

Def. A model diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}_*{}^{\mathrm{op}} \to \mathcal{A}$ s.th. $[\![*]\!] = 1$.

$$
\begin{tikzcd}
* \arrow[d, "p"'] \\
\tau
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \\
{[\![\tau]\!]} \arrow[u, "{[\![p]\!] \in \mathcal{A}([\![\tau]\!], 1)}"']
\end{tikzcd}
$$

Frames and Models

Generate a category $\mathbf{D}$ from basic types τ and labels $i : \tau \to \tau'$.

Kripke frames can then be generalized by

Def. A frame diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}^{\mathrm{op}} \to \mathcal{A}$.

$$
\begin{tikzcd}
\tau \arrow[d, "i"'] & [\![\tau]\!] & \mathcal{A}([\![\tau]\!], 1) \arrow[d, "{[\![i]\!];-}"] & [\![\varphi]\!] \arrow[d, mapsto] \\
\tau' & [\![\tau']\!] \arrow[u, "{[\![i]\!]}"] & \mathcal{A}([\![\tau']\!], 1) & [\![\Diamond_i \varphi]\!]
\end{tikzcd}
$$

Let $\mathbf{D}_*$ be $\mathbf{D}$ with an object $*$ and labels $p : * \to \tau$ added.

Def. A model diagram in $\mathcal{A}$ is a $[\![-]\!] : \mathbf{D}_*{}^{\mathrm{op}} \to \mathcal{A}$ s.th. $[\![*]\!] = 1$.

$$
\begin{tikzcd}
* \arrow[d, "p"'] & 1 \\
\tau & [\![\tau]\!] \arrow[u, "{[\![p]\!] \in \mathcal{A}([\![\tau]\!], 1)}"']
\end{tikzcd}
$$

$\mathbf{D}$ may have more structure: e.g. $\dagger$ for temporal, $\cup$ for dynamic logics.

Interpretation

For propositions of type τ,

$$\begin{aligned}
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] = \overline{[\![\varphi]\!]};[\![\psi]\!], \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!], \\
[\![\varphi \Rightarrow \psi]\!] &= \overline{[\![\varphi]\!]}\backslash[\![\psi]\!], \\
[\![\neg\varphi]\!] &= [\![\varphi \Rightarrow \bot_\tau]\!], \\
[\![\top_\tau]\!] &= \top_{([\![\tau]\!],1)}, \\
[\![\bot_\tau]\!] &= \bot_{([\![\tau]\!],1)}.
\end{aligned}$$

For $i : \tau \to \tau'$, given $[\![\varphi]\!] : [\![\tau]\!] \nrightarrow 1$,

$$\begin{aligned}
[\![\Diamond_i \varphi]\!] &= [\![i]\!];[\![\varphi]\!] : [\![\tau']\!] \nrightarrow 1, \\
[\![\Box_i \varphi]\!] &= [\![i]\!]^\dagger\backslash[\![\varphi]\!] : [\![\tau']\!] \nrightarrow 1.
\end{aligned}$$

Example

Simpson's (1994) semantics in terms of "birelation models":

- A frame is a poset $(X, \leqslant)$ plus $R : X \nrightarrow X$ s.th.

$$
\begin{tikzcd}
X \arrow[r, "R", "\shortmid"{marking}] & X & X & X \arrow[l, "R^\dagger"', "\shortmid"{marking}] \\
X \arrow[u, "\leqslant", "\shortmid"{marking}] \arrow[r, "R"', "\shortmid"{marking}] \arrow[ur, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & X \arrow[u, "\leqslant"', "\shortmid"{marking}] & X \arrow[u, "\leqslant", "\shortmid"{marking}] & X \arrow[u, "\leqslant"', "\shortmid"{marking}] \arrow[l, "R^\dagger", "\shortmid"{marking}] \arrow[ul, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"]
\end{tikzcd}
$$

- Each $[\![p]\!] \subseteq X$ is $\leqslant$-upward closed.

This is to take our allegorical semantics in the allegory of posets and bisimulations.

($[\![p]\!] \subseteq X$ is $\leqslant$-upward closed iff $[\![p]\!] : X \nrightarrow 1$ is a bisimulation.)

Maps of diagrams and bisimulations

Def. A map of diagrams is a map-valued natural transformation.

$$
\begin{tikzcd}
\tau \arrow[d, "i"'] & \llbracket \tau \rrbracket_1 \arrow[r, "\alpha_\tau"] \arrow[dr, phantom, "="] & \llbracket \tau \rrbracket_2 \\
\tau' & \llbracket \tau' \rrbracket_1 \arrow[u, "\llbracket i \rrbracket_1", "|" marking] \arrow[r, "\alpha_{\tau'}"'] & \llbracket \tau' \rrbracket_2 \arrow[u, "\llbracket i \rrbracket_2"', "|" marking]
\end{tikzcd}
$$

Maps of diagrams and bisimulations

Def. A map of diagrams is a map-valued natural transformation.

$$
\begin{tikzcd}
\tau \arrow[d, "i"'] & \llbracket \tau \rrbracket_1 \arrow[r, "\alpha_\tau"] & \llbracket \tau \rrbracket_2 \\
\tau' & \llbracket \tau' \rrbracket_1 \arrow[u, "\llbracket i \rrbracket_1", "|" marking] \arrow[r, "\alpha_{\tau'}"'] \arrow[ur, phantom, "="] & \llbracket \tau' \rrbracket_2 \arrow[u, "\llbracket i \rrbracket_2"', "|" marking]
\end{tikzcd}
$$

Thm.

$$
\begin{tikzcd}
& 1 & \\
\llbracket \tau \rrbracket_1 \arrow[ur, "\llbracket \varphi \rrbracket_1", "|" marking] \arrow[rr, "\alpha_\tau"'] \arrow[rr, phantom, shift left=4ex, "="] & & \llbracket \tau \rrbracket_2 \arrow[ul, "\llbracket \varphi \rrbracket_2"', "|" marking] \\
& 1 \arrow[ul, "x"] \arrow[ur, "y"'] &
\end{tikzcd}
$$

Thm. The correspondence below extends to every $\mathcal{A}$.

$$
\begin{tikzcd}
X \arrow[r, "T", "\mid"{marking}] & Y \\
X \arrow[u, "R", "\mid"{marking}] \arrow[r, "T"', "\mid"{marking}] \arrow[ur, phantom, "\rotatebox{90}{$\subseteq$}"] & Y \arrow[u, "S"', "\mid"{marking}]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X & Z \arrow[l, "f"'] \arrow[r, "g"] & Y \\
X \arrow[u, "R", "\mid"{marking}] & Z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, "U"', "\mid"{marking}] \arrow[ul, phantom, "="] \arrow[ur, phantom, "="] & Y \arrow[u, "S"', "\mid"{marking}]
\end{tikzcd}
$$

Thm. The correspondence below extends to every $\mathcal{A}$.

$$
\begin{tikzcd}
X \arrow[r, "T", "\shortmid" marking] & Y \\
X \arrow[u, "R", "\shortmid" marking] \arrow[r, "T"', "\shortmid" marking] \arrow[ur, phantom, "\rotatebox{90}{$\subseteq$}"] & Y \arrow[u, "S"', "\shortmid" marking]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X & Z \arrow[l, "f"'] \arrow[r, "g"] & Y \\
X \arrow[u, "R", "\shortmid" marking] & Z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, "U"', "\shortmid" marking] \arrow[ul, phantom, "="] \arrow[ur, phantom, "="] & Y \arrow[u, "S"', "\shortmid" marking]
\end{tikzcd}
$$

Def. A bisimulation of diagrams is a span of maps.

Thm. The correspondence below extends to every $\mathcal{A}$.

$$
\begin{tikzcd}
X \arrow[r, "T", "\shortmid"{marking}] & Y \\
X \arrow[u, "R", "\shortmid"{marking}] \arrow[r, "T"', "\shortmid"{marking}] \arrow[ur, phantom, "\subseteq"] & Y \arrow[u, "S"', "\shortmid"{marking}]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X & Z \arrow[l, "f"'] \arrow[r, "g"] & Y \\
X \arrow[u, "R", "\shortmid"{marking}] \arrow[ur, phantom, "="] & Z \arrow[l, "f"] \arrow[r, "g"'] \arrow[u, "U"', "\shortmid"{marking}] \arrow[ur, phantom, "="] & Y \arrow[u, "S"', "\shortmid"{marking}]
\end{tikzcd}
$$

Def. A bisimulation of diagrams is a span of maps.

Thm.

$$
\begin{tikzcd}
 & 1 & \\
{[\![\tau]\!]_1} \arrow[ur, "{[\![\varphi]\!]_1}", "\shortmid"{marking}] & H(\tau) \arrow[l, "\alpha_\tau"] \arrow[r, "\beta_\tau"'] \arrow[u, "H(\varphi)"', "\shortmid"{marking}] & {[\![\tau]\!]_2} \arrow[ul, "{[\![\varphi]\!]_2}"', "\shortmid"{marking}] \\
 & 1 \arrow[ul, "x"] \arrow[ur, "y"'] \arrow[u, dotted, "z"'] &
\end{tikzcd}
$$

Duality and correspondence

For a nice enough $\mathcal{A}$, we have order embeddings

$$\exists_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)),$$

and order-reversing embeddings

$$\forall_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)).$$

Duality and correspondence

For a nice enough $\mathcal{A}$, we have order embeddings

$$\exists_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)),$$

and order-reversing embeddings

$$\forall_{\_^\dagger} : \mathcal{A}(X,Y) \to \mathbf{Pos}(\mathcal{A}(Y,1), \mathcal{A}(X,1)).$$

Thm. In such an $\mathcal{A}$, the condition ${R_1}^\dagger;R_2 \subseteq R_3;{R_4}^\dagger$ corresponds to

$$\Diamond_2\Box_4\varphi \vdash \Box_1\Diamond_3\varphi, \qquad\qquad \Diamond_1\Box_3\varphi \vdash \Box_2\Diamond_4\varphi.$$

Duality and correspondence

For a nice enough $\mathcal{A}$, we have order embeddings

$$\exists_{\_^\dagger} : \mathcal{A}(X, Y) \to \mathbf{Pos}(\mathcal{A}(Y, 1), \mathcal{A}(X, 1)),$$

and order-reversing embeddings

$$\forall_{\_^\dagger} : \mathcal{A}(X, Y) \to \mathbf{Pos}(\mathcal{A}(Y, 1), \mathcal{A}(X, 1)).$$

Thm. In such an $\mathcal{A}$, the condition ${R_1}^\dagger;R_2 \subseteq R_3;{R_4}^\dagger$ corresponds to

$$\Diamond_2 \Box_4 \varphi \vdash \Box_1 \Diamond_3 \varphi, \qquad\qquad \Diamond_1 \Box_3 \varphi \vdash \Box_2 \Diamond_4 \varphi.$$

Indeed, (the intuitionistic version of) the much stronger "calculus for correspondence" (Conradie et al. 2014) is sound in any division $\mathcal{A}$ s.th. $\mathbf{Map}(\mathcal{A})$ is well-pointed.

Standard translation into categorical logic of $\mathbf{Map}(\mathcal{A})$.

$$
\begin{aligned}
(\, x : T \mid \mathrm{tr}(p : \tau)\,) &= (\, x : T \mid Px\,),\\
(\, x : T \mid \mathrm{tr}(\bot : \tau)\,) &= (\, x : T \mid x \neq x\,),\\
(\, x : T \mid \mathrm{tr}(\varphi \wedge \psi : \tau)\,) &= (\, x : T \mid \mathrm{tr}(\varphi : \tau) \wedge \mathrm{tr}(\psi : \tau)\,),\\
(\, x : T \mid \mathrm{tr}(\Box_i \varphi : \tau)\,) &= (\, x : T \mid \forall y : T' \,(R_i x y \Rightarrow \mathrm{tr}(\varphi : \tau')[y/x]\,),\\
(\, x : T \mid \mathrm{tr}(\Diamond_i \varphi : \tau)\,) &= (\, x : T \mid \exists y : T' \,(R_i x y \wedge \mathrm{tr}(\varphi : \tau')[y/x]\,).
\end{aligned}
$$

Logic of the semantics

Since $\exists_{R^\dagger}$ and $\forall_{R^\dagger}$ are left and right adjoints,

$$\frac{\varphi \vdash_\tau \psi}{\Diamond\varphi \vdash_{\tau'} \Diamond\psi} \qquad \frac{\varphi \vdash_\tau \psi}{\Box\varphi \vdash_{\tau'} \Box\psi}$$

$$\Diamond(\varphi \vee \psi) \vdash_{\tau'} \Diamond\varphi \vee \Diamond\psi \qquad \Box\varphi \wedge \Box\psi \vdash_{\tau'} \Box(\varphi \wedge \psi)$$

$$\Diamond\bot_\tau \vdash_{\tau'} \bot_{\tau'} \qquad \top_{\tau'} \vdash_{\tau'} \Box\top_\tau$$

Logic of the semantics

Since $\exists_{R^\dagger}$ and $\forall_{R^\dagger}$ are left and right adjoints,

$$\frac{\varphi \vdash_\tau \psi}{\Diamond\varphi \vdash_{\tau'} \Diamond\psi} \qquad \frac{\varphi \vdash_\tau \psi}{\Box\varphi \vdash_{\tau'} \Box\psi}$$

$$\Diamond(\varphi \vee \psi) \vdash_{\tau'} \Diamond\varphi \vee \Diamond\psi \qquad \Box\varphi \wedge \Box\psi \vdash_{\tau'} \Box(\varphi \wedge \psi)$$

$$\Diamond\bot_\tau \vdash_{\tau'} \bot_{\tau'} \qquad \top_{\tau'} \vdash_{\tau'} \Box\top_\tau$$

The following are sound by the modular law.

$$\Diamond\varphi \wedge \Box\chi \vdash \Diamond(\varphi \wedge \chi)$$

$$(\Diamond\varphi \Rightarrow \Box\psi) \vdash \Box(\varphi \Rightarrow \psi)$$

Logic of the semantics

Since $\exists_{R^\dagger}$ and $\forall_{R^\dagger}$ are left and right adjoints,

$$\frac{\varphi \vdash_\tau \psi}{\Diamond\varphi \vdash_{\tau'} \Diamond\psi} \qquad \frac{\varphi \vdash_\tau \psi}{\Box\varphi \vdash_{\tau'} \Box\psi}$$

$$\Diamond(\varphi \vee \psi) \vdash_{\tau'} \Diamond\varphi \vee \Diamond\psi \qquad \Box\varphi \wedge \Box\psi \vdash_{\tau'} \Box(\varphi \wedge \psi)$$

$$\Diamond\bot_\tau \vdash_{\tau'} \bot_{\tau'} \qquad \top_{\tau'} \vdash_{\tau'} \Box\top_\tau$$

The following are sound by the modular law.

$$\Diamond\varphi \wedge \Box\chi \vdash \Diamond(\varphi \wedge \chi)$$

$$(\Diamond\varphi \Rightarrow \Box\psi) \vdash \Box(\varphi \Rightarrow \psi)$$

This is in fact a typed version of **IK** (the logic of Simpson's (1994) semantics). Call it **tIK**.

Logic of the semantics

Since $\exists_{R^\dagger}$ and $\forall_{R^\dagger}$ are left and right adjoints,

$$\frac{\varphi \vdash_\tau \psi}{\Diamond\varphi \vdash_{\tau'} \Diamond\psi} \qquad \frac{\varphi \vdash_\tau \psi}{\Box\varphi \vdash_{\tau'} \Box\psi}$$

$$\Diamond(\varphi \vee \psi) \vdash_{\tau'} \Diamond\varphi \vee \Diamond\psi \qquad \Box\varphi \wedge \Box\psi \vdash_{\tau'} \Box(\varphi \wedge \psi)$$

$$\Diamond\bot_\tau \vdash_{\tau'} \bot_{\tau'} \qquad \top_{\tau'} \vdash_{\tau'} \Box\top_\tau$$

The following are sound by the modular law.

$$\Diamond\varphi \wedge \Box\chi \vdash \Diamond(\varphi \wedge \chi)$$

$$(\Diamond\varphi \Rightarrow \Box\psi) \vdash \Box(\varphi \Rightarrow \psi)$$

This is in fact a typed version of **IK** (the logic of Simpson's (1994) semantics). Call it **tIK**.

Thm. **tIK** is sound and complete w.r.t. all allegorical semantics.

Future Work

- More on bisimulation theorems. In particular, Hennessy-Milner and van Benthem-type theorems.
- Model-checking.
- More variants of modal logic. E.g. fixed point logic.
- Axiomatization of smaller fragments. E.g. without division structure.
- Axiomatization of particular base logics. E.g. the allegory of fuzzy relations.
- In particular, $\mathbf{Rel}(\mathcal{C})$ as models of quantum theory (Heunen-Tull 2015).
- Diagrammatic methods for the distribution and division structures.