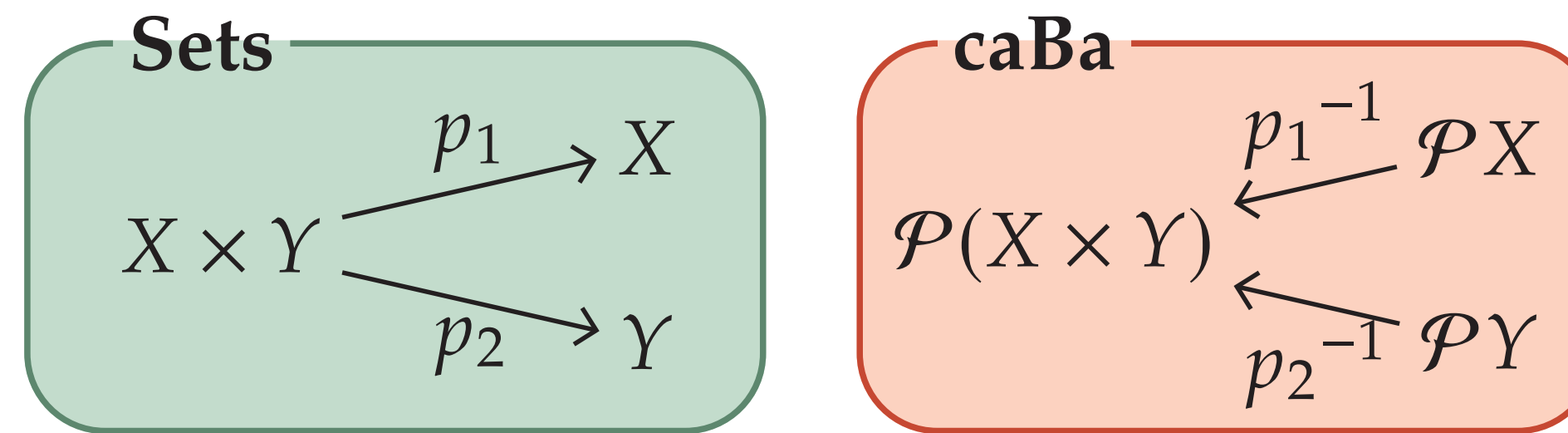


CATS ARE GOOD FOR US!

Category theory studies a given type of structure by studying its “category”, the family of objects of that type and relationships among them. Various constructions are captured by “universal arrows” and “functors” (homomorphisms of cats). E.g.,



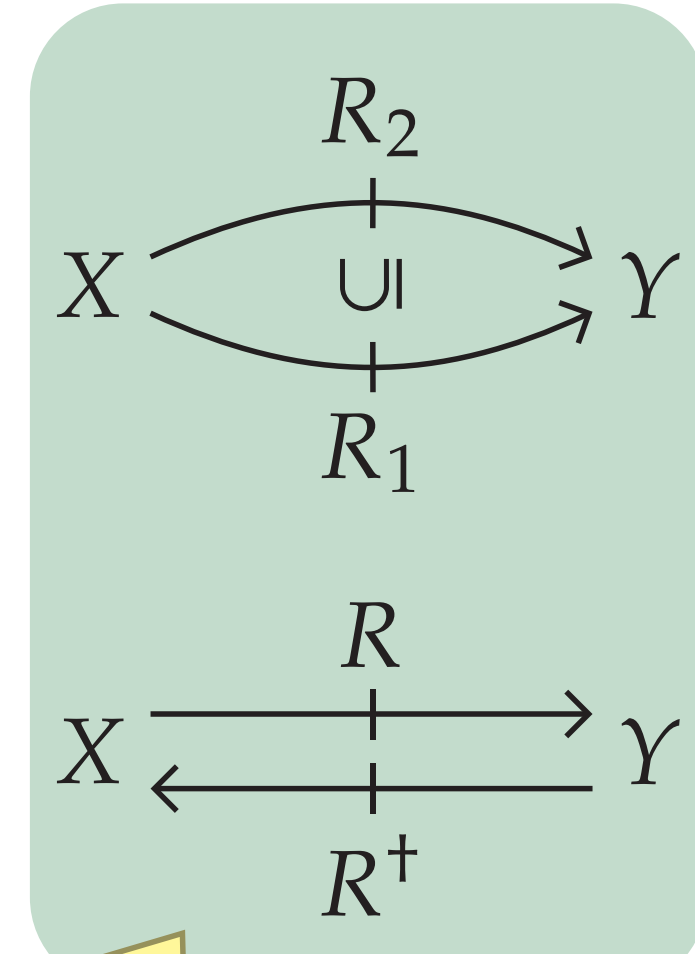
In DEL, one considers a family of models in which one model is updated to another by constructions that model informational processes. This is precisely what category theory is there for!

CATEGORY OF RELATIONS

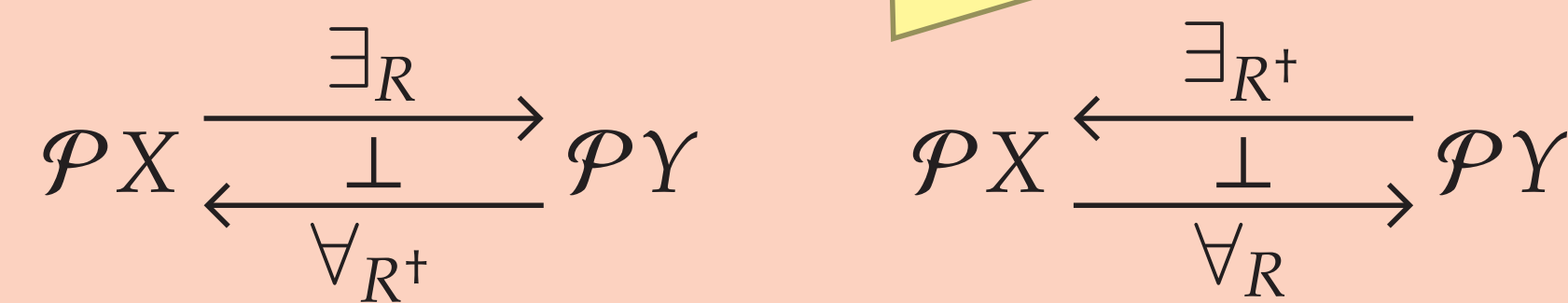
In the cat **Rel** of sets and binary relations, relations are ordered by \subseteq , and each relation R has an opposite R^\dagger . This makes **Rel** a “higher cat” and a self-dual “dagger cat”.

Rel can express e.g. reflexivity of $R : X \rightarrow X$ (i.e. $w = v$ implying wRv) as $1_X \subseteq R$. In particular, **Sets** is the subcat of $f : X \rightarrow Y$ s.th. $1_X \subseteq f^\dagger \circ f$ and $f \circ f^\dagger \subseteq 1_Y$.

In **Sets**, $R \subseteq X_1 \times X_2$ are “tabulated” by $r_i : R \rightarrow X_i$, so that $R = r_2 \circ r_1^\dagger : X_1 \rightarrow X_2$.



(HIGHER) DUALITY OF RELATIONS AND MODALITIES



Given a relation $R : X \rightarrow Y$, define monotone

$$\exists_R(S) = \{v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv\},$$

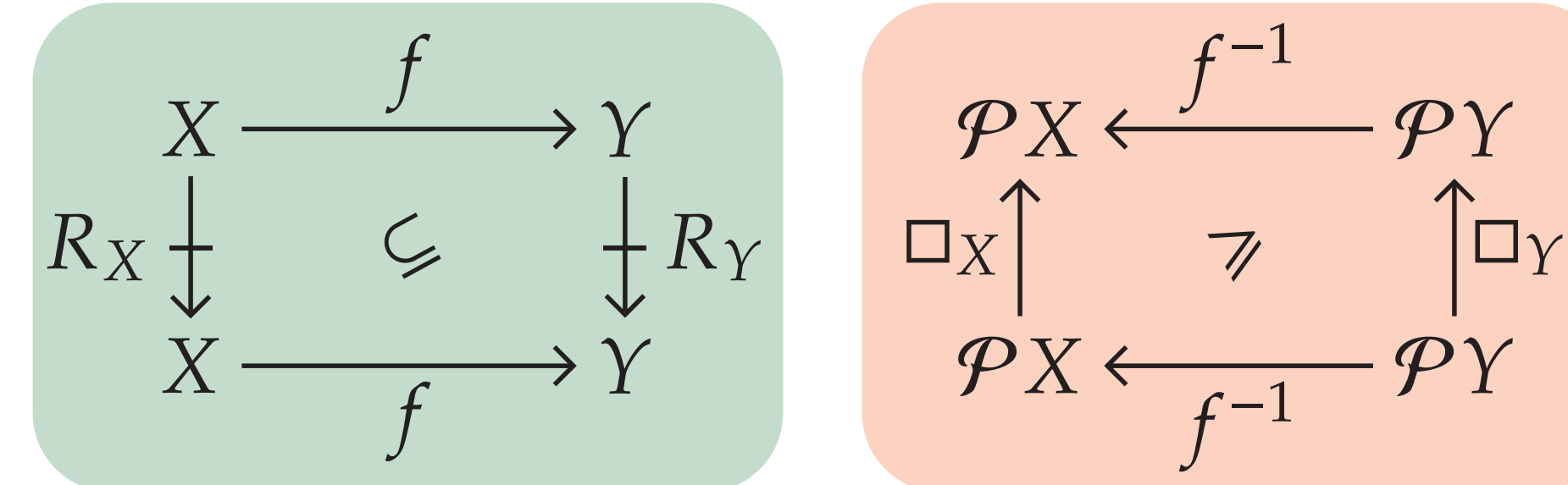
$$\forall_R(S) = \{v \in Y \mid w \in S \text{ for all } w \text{ s.th. } wRv\}.$$

Then $\diamond_R = \exists_{R^\dagger}$ and $\square_R = \forall_{R^\dagger}$ for $R : X \rightarrow X$. Also, $f^{-1} = \exists_{f^\dagger} = \forall_{f^\dagger}$ for functions $f : X \rightarrow Y$.

Maps $h, k : \mathcal{P}X \rightarrow \mathcal{P}Y$ are ordered: $h \leq k$ if $h(S) \subseteq k(S)$ for all $S \subseteq X$. Then a theorem:

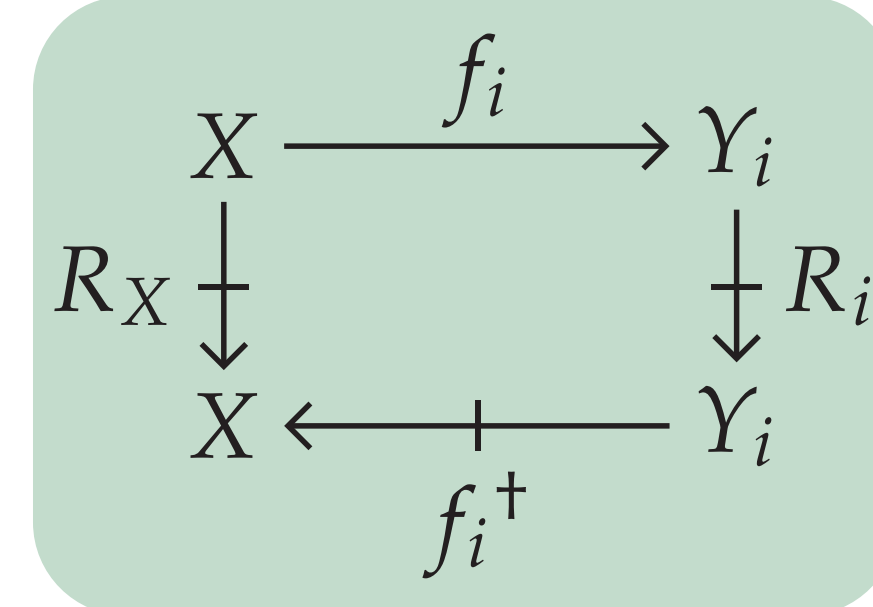
CATEGORIES OF KRIPKE FRAMES

$f : X \rightarrow Y$ between Kripke frames is *monotone* iff



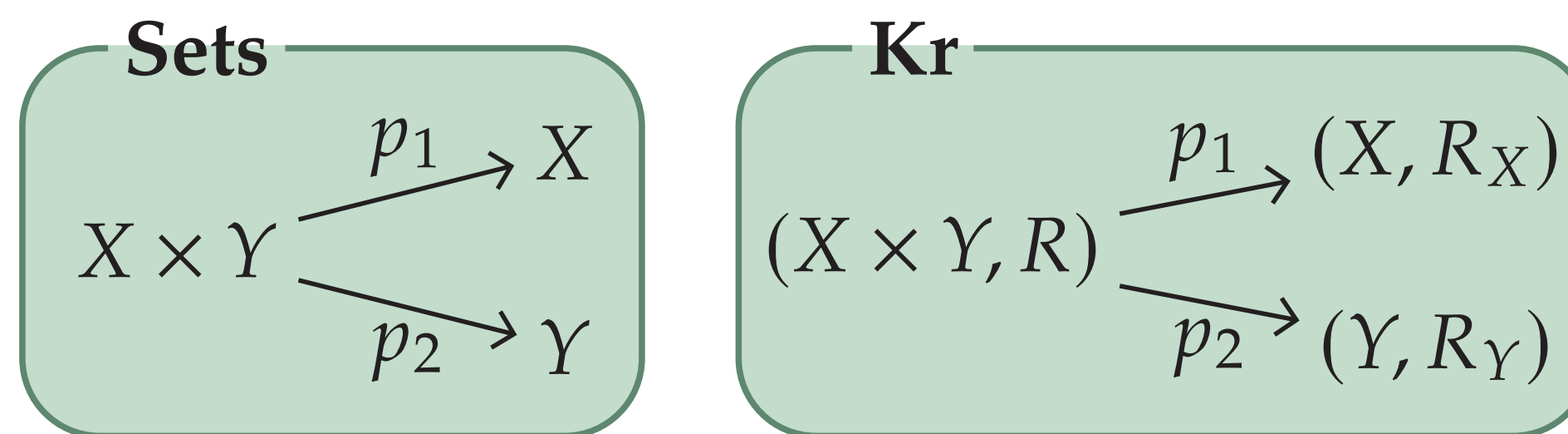
f is a bounded morphism iff “=” holds instead of “ \subseteq ”, “ \leq ”. The cat **Kr_B** of bounded morphisms is “the” cat for static modal logic. But for DEL, the cat **Kr** of monotone maps is equally important.

Kr is “topological over **Sets**”: Given any family of functions $f_i : X \rightarrow Y_i$ to (Y_i, R_i) , it has a unique “initial lift”



$$R_X = \bigcap_i (f_i^\dagger \circ R_i \circ f_i)$$

(the largest relation preserved by all f_i), so that f_i are universal. It follows that every limit or colimit in **Sets** “lifts” to one in **Kr**, on the same set and functions. E.g. products:

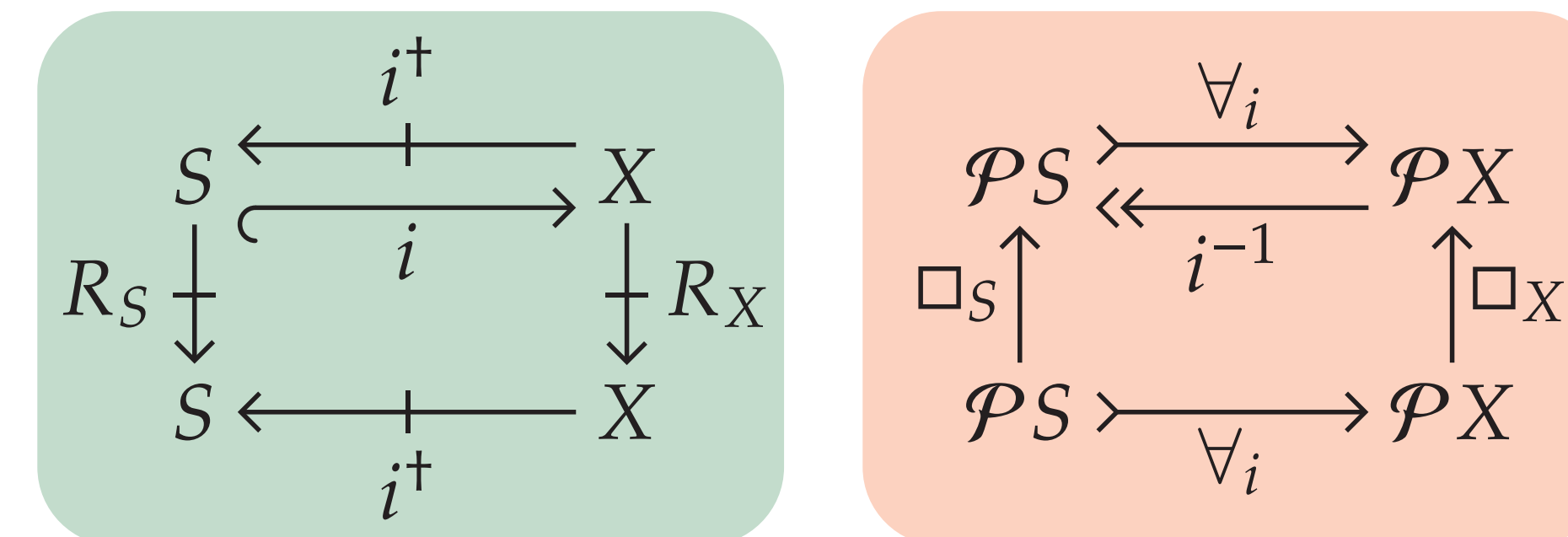


Also, every subset $i : S \hookrightarrow X$ of (X, R_X) lifts to the subframe on S . These constructions, essential for DEL, take place in **Kr** but not **Kr_B**.

SEMANTICS OF DEL, CATEGORICALLY

A submodel is given by the initial lift of $i : S \hookrightarrow X$, i.e. $R_S = i^\dagger \circ R_X \circ i$, with $\llbracket p \rrbracket_S = i^{-1} \llbracket p \rrbracket_X$. With $S = \llbracket \sigma \rrbracket_X$, public announcement of σ is

- $\llbracket [\sigma!] \varphi \rrbracket_X = \forall_i \llbracket \varphi \rrbracket_S$, the modality of relation i^\dagger ($w \in \llbracket [\sigma!] \varphi \rrbracket_X$ iff $v \in \llbracket \varphi \rrbracket_S$ whenever viw).
- In contrast, $\llbracket \sigma \Rightarrow \varphi \rrbracket_X = \forall_i \circ i^{-1} \llbracket \varphi \rrbracket_X$ is also a modality, viz. that of $i \circ i^\dagger$.



Reduction axioms follow immediately:

- $\llbracket [\sigma!] p \rrbracket_X = \forall_i \llbracket p \rrbracket_S = \forall_i \circ i^{-1} \llbracket p \rrbracket_X = \llbracket \sigma \Rightarrow p \rrbracket_X$.
- The dual of $R_S \circ i^\dagger = i^\dagger \circ R_X \circ i \circ i^\dagger$ gives

$$\forall_i \circ \forall_{R_S} \llbracket \varphi \rrbracket_S = \forall_i \circ i^{-1} \circ \forall_{R_X} \circ \forall_i \llbracket \varphi \rrbracket_S$$

$$\llbracket [\sigma!] \square \varphi \rrbracket_X \quad \llbracket \sigma \Rightarrow \square [\sigma!] \varphi \rrbracket_X$$

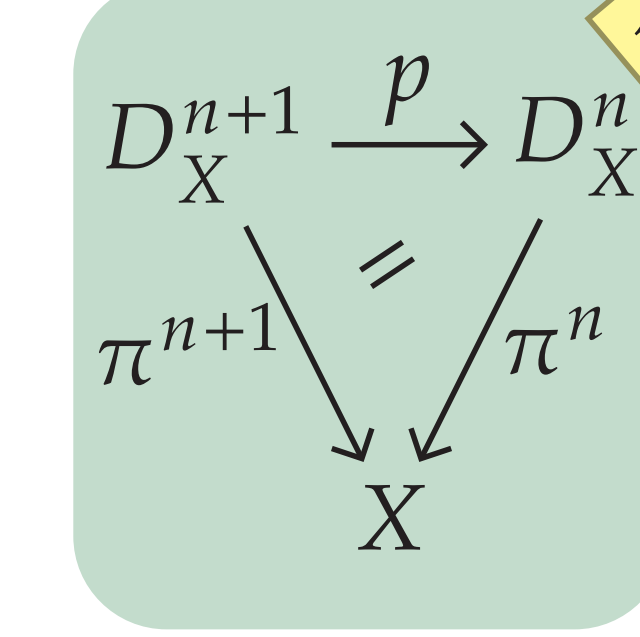
APPLICATION: FIRST-ORDER DEL

Categorical, structural characterization makes various constructions easier to extend, e.g. to use them as modules for combining. One example is to combine first-order and DEL structures, by extending product update to *pullback update*.

APPENDIX: KRIPKE-SHEAF SEMANTICS

Kripke-sheaf semantics models FOL with a map $\pi : D \rightarrow X$ of “possible individuals” to the worlds they live in. Take n -fold products D_X^n in the “slice cat” **Kr**/ (X, R_X) , and then

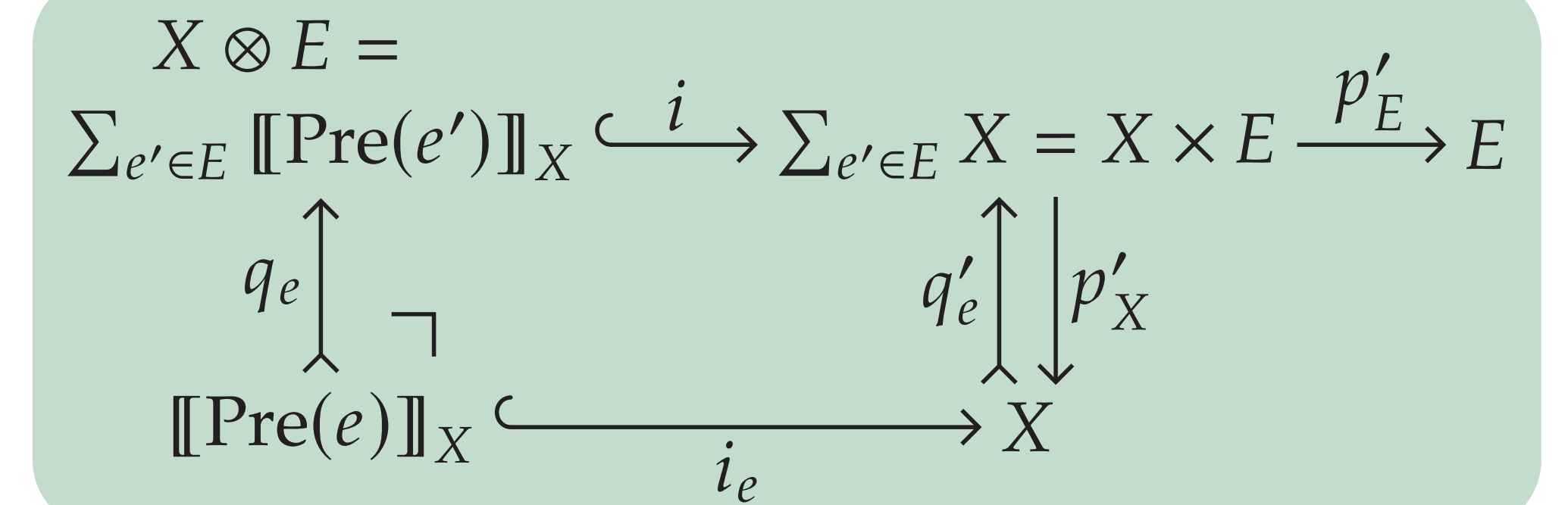
- $\llbracket \bar{x} \mid \varphi \rrbracket \subseteq D_X^n$ interprets an n -ary formula φ in context \bar{x} ,
- $\llbracket \bar{x} \mid \forall y. \varphi \rrbracket = \forall_p \llbracket \bar{x}, y \mid \varphi \rrbracket$.
- But each D_X^n is a Kripke frame,
- so $\llbracket \bar{x} \mid \square \varphi \rrbracket = \forall_{R_{D_X^n}} \llbracket \bar{x} \mid \varphi \rrbracket$.



Every map involved here is a bounded morphism if π is a “Kripke sheaf”. It then means a certain coherence condition that implies

- The simple union **FOK** of FOL and **K** is sound and complete w.r.t. Kripke-sheaf semantics.

Analyzing product update, $R_{X \otimes E}$ is the initial lift of projections $p_X = p'_X \circ i$ and $p_E = p'_E \circ i$ in



- $[E, e]$ is the modality of $R_e = q_e \circ i_e^\dagger = i^\dagger \circ q'_e$.
- $\text{Pre}(e) \Rightarrow -$ is that of $p_X \circ R_e = i_e \circ i_e^\dagger$.

The reduction axiom for \square is by the dual of

$$R_{X \otimes E} \circ R_e = (\bigcup_{e' \in RE'} R_{e'}) \circ R_X \circ i_e \circ i_e^\dagger,$$

$$\forall_{R_e} \circ \forall_{R_{X \otimes E}} = \forall_{i_e} \circ i_e^{-1} \circ \forall_{R_X} \circ \bigcap_{e' \in RE'} \forall_{R_{e'}}.$$

WHY MERELY MONOTONE MAPS?

It is crucial that p_X is *not* a bounded morphism. If it is, $\llbracket \varphi \rrbracket_{X \otimes E} = p_X^{-1} \llbracket \varphi \rrbracket_X$ and $[E, e]$ boils down to $\text{Pre}(e) \Rightarrow -$ (so events teach nothing to agents).

Also, monotone maps can tabulate any relation between Kripke frames, but bounded morphisms can only tabulate bisimulations.

The key is $p_X^* : \mathbf{Kr}/X \rightarrow \mathbf{Kr}/X \otimes E$, the pullback functor along the canonical map $p_X : X \otimes E \rightarrow X$. This pulls back a Kripke-sheaf model to another,

and each $p_X^* D_X^n$ is the product update of D_X^n . This validates the usual reduction axioms and

$[E, e] \forall y. \varphi \equiv \forall y. [E, e] \varphi$ for \forall , making **FOK** plus them sound and complete.

MUCH MORE CAN BE DONE

- Categorical analysis of more vocabularies (e.g. common knowledge), structures (e.g. probability), and types of logic (e.g. intuitionistic-based).
- Theory and characterization of reducibility.
- p_X^* as a map of “toposes” of Kripke sheaves.
- Duality theory with a “syntactic cat” of DEL.