Categories for Dynamic Epistemic Logic

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Cats are Good for Us!

Category theory studies a given type of structure by studying its "category", the family of objects of that type and relationships among them. Various constructions are captured by "universal arrows" and "functors" (homomorphisms of cats). E.g.,

Sets

$$\begin{tikzcd} & X \\ X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & Y \end{tikzcd}$$

caBa

$$\begin{tikzcd} & \mathcal{P}X \arrow[dl, "p_1^{-1}"'] \\ \mathcal{P}(X \times Y) & \\ & \mathcal{P}Y \arrow[ul, "p_2^{-1}"] \end{tikzcd}$$

In DEL, one considers a family of models in which one model is updated to another by constructions that model informational processes. This is precisely what category theory is there for!

Category of Relations

In the cat **Rel** of sets and binary relations, relations are ordered by $\subseteq$, and each relation R has an opposite $R^\dagger$. This makes **Rel** a "higher cat" and a self-dual "dagger cat".

Rel can express e.g. reflexivity of $R : X \nrightarrow X$ (i.e. $w = v$ implying wRv) as $1_X \subseteq R$. In particular, **Sets** is the subcat of $f : X \nrightarrow Y$ s.th. $1_X \subseteq f^\dagger \circ f$ and $f \circ f^\dagger \subseteq 1_Y$.

In **Sets**, $R \subseteq X_1 \times X_2$ are "tabulated" by $r_i : R \to X_i$, so that $R = r_2 \circ r_1{}^\dagger : X_1 \nrightarrow X_2$.

$$\begin{tikzcd} X \arrow[r, bend left, "R_2"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\cup\!\shortmid"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, shift left, "R"] & Y \arrow[l, shift left, "R^\dagger"] \end{tikzcd}$$

dual

Categories of Kripke Frames

$f : X \to Y$ between Kripke frames is *monotone* iff

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "R_X"'] \arrow[dr, phantom, "\subseteq"] & Y \arrow[d, "R_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[dr, phantom, "\geqslant"] & \mathcal{P}Y \arrow[l, "f^{-1}"'] \\ \mathcal{P}X \arrow[u, "\Box_X"] & \mathcal{P}Y \arrow[l, "f^{-1}"] \arrow[u, "\Box_Y"'] \end{tikzcd}$$

f is a bounded morphism iff "=" holds instead of "$\subseteq$", "$\leqslant$". The cat $\mathbf{Kr_B}$ of bounded morphisms is "the" cat for static modal logic. But for DEL, the cat **Kr** of monotone maps is equally important.

Kr is "topological over **Sets**": Given any family of functions $f_i : X \to Y_i$ to (Y_i, R_i), it has a unique "initial lift"

$$R_X = \textstyle\bigcap_i (f_i{}^\dagger \circ R_i \circ f_i)$$

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "R_X"'] & Y_i \arrow[d, "R_i"] \\ X & Y_i \arrow[l, "f_i{}^\dagger"] \end{tikzcd}$$

(the largest relation preserved by all f_i), so that f_i are universal. It follows that every limit or colimit in **Sets** "lifts" to one in **Kr**, on the same set and functions. E.g. products:

Sets

$$\begin{tikzcd} & X \\ X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & Y \end{tikzcd}$$

Kr

$$\begin{tikzcd} & (X, R_X) \\ (X \times Y, R) \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & (Y, R_Y) \end{tikzcd}$$

Also, every subset $i : S \hookrightarrow X$ of (X, R_X) lifts to the subframe on S. These constructions, essential for DEL, take place in **Kr** but not $\mathbf{Kr_B}$.

(Higher) Duality of Relations and Modalities

$$\begin{tikzcd} \mathcal{P}X \arrow[r, shift left=2, "\exists_R"] \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift left=2, "\forall_{R^\dagger}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}X \arrow[r, phantom, "\perp"] & \mathcal{P}Y \arrow[l, shift right=2, "\exists_{R^\dagger}"'] \arrow[l, shift left=2, "\forall_R"] \end{tikzcd}$$

Given a relation $R : X \nrightarrow Y$, define monotone

$$\exists_R(S) = \{\, v \in Y \mid w \in S \text{ for some } w \text{ s.th. } wRv \,\},$$
$$\forall_R(S) = \{\, v \in Y \mid w \in S \text{ for all } w \text{ s.th. } wRv \,\}.$$

Then $\Diamond_R = \exists_{R^\dagger}$ and $\Box_R = \forall_{R^\dagger}$ for $R : X \nrightarrow X$. Also, $f^{-1} = \exists_{f^\dagger} = \forall_{f^\dagger}$ for functions $f : X \to Y$.

Maps $h, k : \mathcal{P}X \to \mathcal{P}Y$ are ordered: $h \leqslant k$ if $h(S) \subseteq k(S)$ for all $S \subseteq X$. Then a theorem:

- $\exists_-$ gives an order iso from the relations $R : X \nrightarrow Y$ to the all-join-preserving $h : \mathcal{P}X \to \mathcal{P}Y$.
$$R_1 \subseteq R_2 \iff \exists_{R_1} \leqslant \exists_{R_2} \iff \exists_{R_1{}^\dagger} \leqslant \exists_{R_2{}^\dagger}.$$
- $\forall_-$ gives an order-reversing iso from $R : X \nrightarrow Y$ to the all-meet-preserving $h : \mathcal{P}X \to \mathcal{P}Y$.
$$R_1 \subseteq R_2 \iff \forall_{R_2} \leqslant \forall_{R_1} \iff \forall_{R_2{}^\dagger} \leqslant \forall_{R_1{}^\dagger}.$$

In short, $\Diamond_R = \exists_{R^\dagger}$ and $\Box_R = \forall_{R^\dagger}$ are dual to R.

Correspondence results follow, e.g,

- R is reflexive iff $1_X \subseteq R$ iff $\Box_R \leqslant 1_{\mathcal{P}X}$ iff $\Box\varphi \vdash \varphi$.
- R is transitive iff $R \circ R \subseteq R$ iff $\Box_R \leqslant \Box_R \circ \Box_R$.

Semantics of DEL, Categorically

A submodel is given by the initial lift of $i : S \hookrightarrow X$, i.e. $R_S = i^\dagger \circ R_X \circ i$, with $[\![p]\!]_S = i^{-1}[\![p]\!]_X$. With $S = [\![\sigma]\!]_X$, public announcement of σ is

- $[\![[\sigma!]\varphi]\!]_X = \forall_i [\![\varphi]\!]_S$, the modality of relation $i^\dagger$ ($w \in [\![[\sigma!]\varphi]\!]_X$ iff $v \in [\![\varphi]\!]_S$ whenever $v i w$).
- In contrast, $[\![\sigma \Rightarrow \varphi]\!]_X = \forall_i \circ i^{-1}[\![\varphi]\!]_X$ is also a modality, viz. that of $i \circ i^\dagger$.

$$\begin{tikzcd} S \arrow[d, "R_S"'] \arrow[r, hook, shift right, "i"'] & X \arrow[l, shift right, "i^\dagger"'] \arrow[d, "R_X"] \\ S & X \arrow[l, "i^\dagger"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}S \arrow[r, shift left=2, "\forall_i"] & \mathcal{P}X \arrow[l, "i^{-1}"] \\ \mathcal{P}S \arrow[u, "\Box_S"] \arrow[r, "\forall_i"'] & \mathcal{P}X \arrow[u, "\Box_X"'] \end{tikzcd}$$

Reduction axioms follow immediately:

- $[\![[\sigma!]p]\!]_X = \forall_i[\![p]\!]_S = \forall_i \circ i^{-1}[\![p]\!]_X = [\![\sigma \Rightarrow p]\!]_X$.
- The dual of $R_S \circ i^\dagger = i^\dagger \circ R_X \circ i \circ i^\dagger$ gives

$$\begin{array}{ccc} \forall_i \circ \forall_{R_S{}^\dagger}[\![\varphi]\!]_S & = & \forall_i \circ i^{-1} \circ \forall_{R_X{}^\dagger} \circ \forall_i[\![\varphi]\!]_S \\ \| & & \| \\ [\![[\sigma!]\Box\varphi]\!]_X & & [\![\sigma \Rightarrow \Box[\sigma!]\varphi]\!]_X. \end{array}$$

Analyzing product update, $R_{X\otimes E}$ is the initial lift of projections $p_X = p'_X \circ i$ and $p_E = p'_E \circ i$ in

$$\begin{tikzcd} X \otimes E = \sum_{e' \in E}[\![\mathrm{Pre}(e')]\!]_X \arrow[r, hook, "i"] & \sum_{e' \in E} X = X \times E \arrow[r, "p'_E"] \arrow[d, shift left, "p'_X"] & E \\ {[\![\mathrm{Pre}(e)]\!]_X} \arrow[u, "q_e"] \arrow[r, hook, "i_e"'] \arrow[ur, phantom, "\lrcorner", very near start] & X \arrow[u, shift left, "q'_e"] & \end{tikzcd}$$

- $[E, e]$ is the modality of $R_e = q_e \circ i_e{}^\dagger = i^\dagger \circ q'_e$.
- $\mathrm{Pre}(e) \Rightarrow -$ is that of $p_X \circ R_e = i_e \circ i_e{}^\dagger$.

The reduction axiom for $\Box$ is by the dual of

$$R_{X\otimes E} \circ R_e = (\textstyle\bigcup_{eR_Ee'} R_{e'}) \circ R_X \circ i_e \circ i_e{}^\dagger,$$
$$\forall_{R_e{}^\dagger} \circ \forall_{R_{X\otimes E}{}^\dagger} = \forall_{i_e} \circ i_e{}^{-1} \circ \forall_{R_X{}^\dagger} \circ \textstyle\bigcap_{eR_Ee'} \forall_{R_{e'}{}^\dagger}.$$

Why Merely Monotone Maps?

It is crucial that p_X is *not* a bounded morphism. If it is, $[\![\varphi]\!]_{X\otimes E} = p^{-1}[\![\varphi]\!]_X$ and $[E, e]$ boils down to $\mathrm{Pre}(e) \Rightarrow -$ (so events teach nothing to agents).

Also, monotone maps can tabulate any relation between Kripke frames, but bounded morphisms can only tabulate bisimulations.

Application: First-Order DEL

Categorical, structural characterization makes various constructions easier to extend, e.g. to use them as modules for combining. One example is to combine first-order and DEL structures, by extending product update to *pullback update*.

The key is $p_X{}^* : \mathbf{Kr}/X \to \mathbf{Kr}/X\otimes E$, the pullback functor along the canonical map $p_X : X \otimes E \to X$. This pulls back a Kripke-sheaf model to another, and each $p_X{}^* D^n_X$ is the product update of D^n_X. This validates the usual reduction axioms and

$$[E, e]\forall y.\varphi \equiv \forall y.[E, e]\varphi$$

for $\forall$, making **FOK** plus them sound and complete.

$$\begin{tikzcd} D^m_X \otimes E \arrow[r] \arrow[dr] \arrow[dd] & D^m_X \arrow[dr] \arrow[dd] & \\ & D^n_X \otimes E \arrow[r] \arrow[dl] & D^n_X \arrow[dl] \\ X \otimes E \arrow[r, "p_X"'] & X & \end{tikzcd}$$

Appendix: Kripke-Sheaf Semantics

Kripke-sheaf semantics models FOL with a map $\pi : D \to X$ of "possible individuals" to the worlds they live in. Take n-fold products D^n_X in the "slice cat" $\mathbf{Kr}/(X, R_X)$, and then

- $[\![\, \bar{x} \mid \varphi \,]\!] \subseteq D^n_X$ interprets an n-ary formula φ in context $\bar{x}$,
- $[\![\, \bar{x} \mid \forall y.\varphi \,]\!] = \forall_p[\![\, \bar{x}, y \mid \varphi \,]\!]$.

But each D^n_X is a Kripke frame,

- so $[\![\, \bar{x} \mid \Box\varphi \,]\!] = \forall_{R_{D^n_X}{}^\dagger}[\![\, \bar{x} \mid \varphi \,]\!]$.

$$\begin{tikzcd} D^{n+1}_X \arrow[r, "p"] \arrow[dr, "\pi^{n+1}"'] \arrow[r, phantom, shift right=4, "="] & D^n_X \arrow[d, "\pi^n"] \\ & X \end{tikzcd}$$

pullback

Every map involved here is a bounded morphism if π is a "Kripke sheaf". It then means a certain coherence condition that implies

- The simple union **FOK** of FOL and **K** is sound and complete w.r.t. Kripke-sheaf semantics.

Much More can be Done

- Categorical analysis of more vocabularies (e.g. common knowledge), structures (e.g. probability), and types of logic (e.g. intuitionistic-based).
- Theory and characterization of reducibility.
- $p_X{}^*$ as a map of "toposes" of Kripke sheaves.
- Duality theory with a "syntactic cat" of DEL.