For this talk

**Definition**

For any subset $S$ of $\mathbb{Z}$ the ring of integer valued polynomials on $S$ is defined to be $\text{Int}(S) = \{ f(x) \in \mathbb{Q}[x] \mid f(S) \subseteq \mathbb{Z} \}$.
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The formal definition is for a domain $D$ and field of fractions $K$.

**Definition**

For any subset $S$ of $D$ the ring of integer valued polynomials on $S$ is defined to be $\text{Int}(S, D) = \{ f(x) \in K[x] \mid f(S) \subseteq D \}$. 
What are Integer Valued Polynomials?

Let's start with $\text{Int}(\mathbb{Z})$ and find some examples:

- $25x^5 - 13x^3 + 7x - 23$
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- Degree 2, \( \frac{x(x-1)}{2} \)

- Degree 3, \( \frac{x(x-1)(x-2)}{2 \cdot 3} \)
What are Integer Valued Polynomials?

In general, for degree $n$:

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

**Theorem**

A polynomial is integer valued on $\mathbb{Z}$ if and only if it can be written as a $\mathbb{Z}$-linear combination of the polynomials

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-k+1)}{n!},$$

for $n = 0, 1, 2, \ldots$.
Today’s Plan

We will go over the following:

- Bases and IVPs on subsets of the integers.
- \( p \)-orderings, \( p \)-sequences and invariants of \( \text{Int}(S) \).
- Multivariable and homogeneous case.
Int(S) is a Ring

- Most of the axioms follow from $\mathbb{Q}[x]$ being a ring.
- Int(S) is also closed under addition and multiplication.
- Int(S) is a $\mathbb{Z}$-module.
**Int(S) is a Module**

**Definition**

An \( R \)-module \( M \), over the ring \( R \) consist of an abelian group \((M, +)\) and an operation \( R \times M \rightarrow M \) (scalar multiplication). For all \( r, s \in R, x, y \in M \) we have

1. \( r(x + y) = rx + ry \).
2. \( (r + s)x = rx + sx \).
3. \( (rs)(x) = r(sx) \).
4. \( 1_Rx = x \).
Int($S$) is a $\mathbb{Z}$-module

In this case $R = \mathbb{Z}$, we want for all $m, n \in \mathbb{Z}$ and $f(x), g(x) \in \text{Int}(S)$:

1. $m(f(x) + g(x)) = m \cdot f(x) + m \cdot g(x)$.

2. $(m + n)f(x) = m \cdot f(x) + n \cdot f(x)$.

3. $(mn)f(x) = m(n \cdot f(x))$.

4. $1 \cdot f(x) = f(x)$.

Multiply $f(x)$ an IVP by an integer $n$ will preserve its integer valued property.
A basis $\mathcal{B}$ of the $R$-module $\mathbb{B}$ is said to be a regular basis if it is formed by one and only one polynomial of each degree.

- A regular basis for $\text{Int}(\mathbb{Z})$ is $\{1\} \cup \left\{ \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \right\}_{n \geq 1}$. 
What if $S \subset \mathbb{Z}$

We will look at $\text{Int}(S)$ for $S \subset \mathbb{Z}$, to motivate why we need better tools to find bases. Here are examples of sets we can consider:

- Even/Odd integers
- Prime numbers
- Fibonacci Numbers
- Sum of $\ell$ $d$-th powers

$$x = x_1^d + x_2^d + \cdots + x_\ell^d$$
Even/Odd Integers

For $S = 2 \mathbb{Z}$, a basis for $\text{Int}(S)$ is made of the polynomials $\left\{ \frac{x}{2^n} \right\}$

$\left\{ 1, x^2, x(x-2)^8, x(x-2)(x-4)^48, \ldots \right\}$

For $S = 1 + 2 \mathbb{Z}$, a basis for $\text{Int}(S)$ is made of the polynomials $\left\{ \frac{(x-1)}{2^n} \right\}$

$\left\{ 1, (x-1)^2, (x-1)(x-3)^8, (x-1)(x-3)(x-5)^48, \ldots \right\}$
Even/Odd Integers

- For $S = 2\mathbb{Z}$, a basis for $\text{Int}(S)$ is made of the polynomials $\left(\frac{x}{2}\right)^n$

  \[
  \left\{1, \frac{x}{2}, \frac{x(x - 2)}{8}, \frac{x(x - 2)(x - 4)}{48}, \ldots \right\}
  \]
Even/Odd Integers

- For $S = 2\mathbb{Z}$, a basis for $\text{Int}(S)$ is made of the polynomials $\binom{x/2}{n}$

\[ \left\{ 1, \frac{x}{2}, \frac{x(x - 2)}{8}, \frac{x(x - 2)(x - 4)}{48}, \ldots \right\} \]

- For $S = 1 + 2\mathbb{Z}$, a basis for $\text{Int}(S)$ is made of the polynomials $\binom{(x-1)/2}{n}$

\[ \left\{ 1, \frac{(x - 1)}{2}, \frac{(x - 1)(x - 3)}{8}, \frac{(x - 1)(x - 3)(x - 5)}{48}, \ldots \right\} \]
The beginning of a basis for $\text{Int}(\mathbb{P})$:

\[
 f_0 = 1, \quad f_1 = (x - 1), \quad f_2 = \frac{(x - 1)(x - 2)}{2},
\]
\[
 f_3 = \frac{(x - 1)(x - 2)(x - 3)}{24}, \quad f_4 = \frac{(x - 1)^2(x - 2)(x - 3)}{48},
\]
\[
 f_5 = \frac{(x - 1)(x - 2)(x - 3)(x - 5)(x - 79)}{5760}, \ldots
\]

and $f_3(4) = \frac{1}{4}, \ f_4(4) = \frac{3}{4}$ and $f_5(4) = \frac{5}{64}$. 

We will go over some work that Bhargava did during his undergraduate degree and define

- $p$-orderings
- $p$-sequences.

For the next part of the presentation we will work locally at a prime $p$. 

Source of Image: https://opc.mfo.de/detail?photo_id=7108
A Game Called $p$-orderings

Fix a prime $p$.

**Definition (Bhargava)**

A $p$-ordering of $S$ a subset of $\mathbb{Z}$ is a sequence $(a_n)_{n \geq 0}$, such that $a_0$ is arbitrarily chosen and for each $n > 0$, $a_n \in S$ is chosen to minimize

$$
\nu_p((a_0 - a_n) \cdots (a_{n-1} - a_n)).
$$

where

$$
\nu_p(n) = \begin{cases} 
\max\{\nu \in \mathbb{N} : p^{\nu} | n\} & n \geq 0 \\
\infty & n = 0
\end{cases}
$$
Let \( p = 2 \) and \( S = \{0, 1, 2, 3, 4\} \)
Let $p = 2$ and $S = \{0, 1, 2, 3, 4\}$

- $a_0 = 0$
Let $p = 2$ and $S = \{0, 1, 2, 3, 4\}$

- $a_0 = 0$
- for $a_1$ we want to minimize the power of 2 dividing $(0 - a_1)$
  take any odd number, $a_1 = 1$
Let $p = 2$ and $S = \{0, 1, 2, 3, 4\}$

- $a_0 = 0$
- for $a_1$ we want to minimize the power of 2 dividing $(0 - a_1)$ take any odd number, $a_1 = 1$
- $a_2 \quad (0 - a_2)(1 - a_2)$ pick 2 or 3, $a_2 = 2$
Let \( p = 2 \) and \( S = \{0, 1, 2, 3, 4\} \)

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- \( a_3 \) \((0 - a_3)(1 - a_3)(2 - a_3)\), need to pick an odd value \( a_3 = 3 \)
Let $p = 2$ and $S = \{0, 1, 2, 3, 4\}$

- $a_0 = 0$
- for $a_1$ we want to minimize the power of 2 dividing $(0 - a_1)$ take any odd number, $a_1 = 1$
- $a_2 \quad (0 - a_2)(1 - a_2)$ pick 2 or 3, $a_2 = 2$
- $a_3 \quad (0 - a_3)(1 - a_3)(2 - a_3)$, need to pick an odd value $a_3 = 3$
- $a_4 = 4$.

Our $p$-ordering of $S$ is $\{0, 1, 2, 3, 4\}$.
Proposition (Bhargava)

The natural ordering of $\mathbb{Z}_{\geq 0}$ with $a_i = i$ is a $p$-ordering of $\mathbb{Z}$ for all primes $p$.

Are $p$-orderings unique?
In General

Proposition (Bhargava)

The natural ordering of $\mathbb{Z} \geq 0$ with $a_i = i$ is a $p$-ordering of $\mathbb{Z}$ for all primes $p$.

Are $p$-orderings unique? No, we made some choices in the previous example.
Definition (Bhargava)

Given \((a_n)_{n \geq 0}\) a \(p\)-ordering of \(S\) a subset of \(\mathbb{Z}\) with \(\alpha_0 = 0\) and \(\alpha_n(S, p) = \nu_p((a_0 - a_n) \cdots (a_{n-1} - a_n))\), \(\{\alpha_n(S, p)\}\) is the associated \(p\)-sequence of \(S\).

A nice property for a set \(S\) is that the \(p\)-sequence is independent of the choice of \(p\)-ordering.

Example : \(\{\alpha_n(\mathbb{Z}, p)\} = \{\nu_p(n!)\}\).
A More Interesting Version of the Game

Let $p = 2$ and $S = \{0, 1, 8, 27, 64, 125\}$
A More Interesting Version of the Game

Let \( p = 2 \) and \( S = \{0, 1, 8, 27, 64, 125\} \)
- \( a_0 = 0 \) and \( \alpha_0 = 0 \)
A More Interesting Version of the Game

Let $p = 2$ and $S = \{0, 1, 8, 27, 64, 125\}$

- $a_0 = 0$ and $\alpha_0 = 0$
- for $a_1$ we want to minimize the power of 2 dividing $(0 - a_1)$
  take any odd number, $a_1 = 1$ and $\alpha_1 = 0$
A More Interesting Version of the Game

Let \( p = 2 \) and \( S = \{0, 1, 8, 27, 64, 125\} \)

- \( a_0 = 0 \) and \( \alpha_0 = 0 \)
- for \( a_1 \) we want to minimize the power of 2 dividing \((0 - a_1)\)
  take any odd number, \( a_1 = 1 \) and \( \alpha_1 = 0 \)
- \( a_2 \) \((0 - a_2)(1 - a_2)\), \( a_2 = 27 \) is the best choice and \( \alpha_2 = 1 \).
A More Interesting Version of the Game

Let $p = 2$ and $S = \{0, 1, 8, 27, 64, 125\}$

- $a_0 = 0$ and $\alpha_0 = 0$
- for $a_1$ we want to minimize the power of 2 dividing $(0 - a_1)$
  take any odd number, $a_1 = 1$ and $\alpha_1 = 0$
- $a_2 \ (0 - a_2)(1 - a_2), \ a_2 = 27$ is the best choice and $\alpha_2 = 1$.
- $a_3 \ (0 - a_3)(1 - a_3)(27 - a_3), \ we \ can \ choose \ between \ 8 \ and \ 125, \ a_3 = 8 \ and \ \alpha_3 = 3$. 
A More Interesting Version of the Game

Let \( p = 2 \) and \( S = \{0, 1, 8, 27, 64, 125\} \)

- \( a_0 = 0 \) and \( \alpha_0 = 0 \)
- for \( a_1 \) we want to minimize the power of 2 dividing \((0 - a_1)\)
take any odd number, \( a_1 = 1 \) and \( \alpha_1 = 0 \)
- \( a_2 \) \((0-a_2)(1-a_2)\), \( a_2 = 27 \) is the best choice and \( \alpha_2 = 1 \).
- \( a_3 \) \((0-a_3)(1-a_3)(27-a_3)\), we can choose between 8 and 125, \( a_3 = 8 \) and \( \alpha_3 = 3 \).
- \( a_4 \) \((0-a_4)(1-a_4)(27-a_4)(8-a_4)\) start checking with 125, since we hope for \( \alpha < 6 \), we obtain \( a_4 = 125 \) and \( \alpha_4 = 3 \)
A More Interesting Version of the Game

Let $p = 2$ and $S = \{0, 1, 8, 27, 64, 125\}$

- $a_0 = 0$ and $\alpha_0 = 0$
- for $a_1$ we want to minimize the power of 2 dividing $(0 - a_1)$
  take any odd number, $a_1 = 1$ and $\alpha_1 = 0$
- $a_2 \ (0 - a_2)(1 - a_2)$, $a_2 = 27$ is the best choice and $\alpha_2 = 1$.
- $a_3 \ (0 - a_3)(1 - a_3)(27 - a_3)$, we can choose between 8 and 125, $a_3 = 8$ and $\alpha_3 = 3$.
- $a_4 \ (0 - a_4)(1 - a_4)(27 - a_4)(8 - a_4)$ start checking with 125, since we hope for $\alpha < 6$, we obtain $a_4 = 125$ and $\alpha_4 = 3$
- $a_5 = 64$ and $(0 - 64)(1 - 64)(27 - 64)(8 - 64)(125 - 64) = 2^6(-63)(-37)(-56)(61)$ and $\alpha_5 = 9$.

Our $p$-ordering of $S$ is $\{0, 1, 27, 8, 125, 64\}$ and the $p$-sequence is $\{0, 0, 1, 3, 3, 9\}$.
Factorial Function

The factorial function is very important when working with IVPs. It was in the denominators of the basis elements of $\text{Int}(\mathbb{Z})$ and in the $p$-sequence of $\mathbb{Z}$.

What is the factorial function when working with $S$ not $\mathbb{Z}$?

**Proposition**

- For any non-negative $k$ and $\ell$, $(k + \ell)!$ is still a multiple of $k!\ell!$.
- Let $f$ be a primitive polynomial with integer coefficients of degree $k$, and let $d(\mathbb{Z}, f) = \gcd\{f(a) \mid a \in \mathbb{Z}\}$. Then $d(\mathbb{Z}, f)$ divides $k!$.

This is called the **fixed divisor** of $f$. 
Definition (Bhargava)

Let $S$ be any subset of $\mathbb{Z}$. the factorial function on $S$ denoted $k!_S$ is defined by

$$k!_S = \prod_p p^{\alpha_k(S,p)}.$$ 

This definition preserves many properties of the factorial function on $\mathbb{Z}$:

- For any non-negative $k$ and $\ell$, $(k + \ell)!_S$ is still a multiple of $k!_S \ell!_S$.
- Let $f$ be a primitive polynomial of degree $k$, and let $d(S, f) = \gcd\{f(a) \mid a \in S\}$. Then $d(S, f)$ divides $k!_S$. 

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We know that $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$, we use the generalized factorial instead:

\[
\alpha_n(\mathbb{Z}, 2) = \{0, 0, 1, 1, 3, 3, 4, 4, 7, 8, \ldots \}
\]
\[
\alpha_n(\mathbb{Z}, 3) = \{0, 0, 0, 1, 1, 1, 2, 2, 2, 4, \ldots \}
\]
\[
\alpha_n(\mathbb{Z}, 5) = \{0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, \ldots \}
\]

\[
6! = 2^4 \cdot 3^2 \cdot 5
\]
Generalized Factorial is the Same on \( \mathbb{Z} \)

We know that \( 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \), we use the generalized factorial instead:

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\alpha_n(\mathbb{Z}, 2) = \{0, 0, 1, 1, 3, 3, 4, 4, 7, 8, \ldots\}
\]

\[
\alpha_n(\mathbb{Z}, 3) = \{0, 0, 0, 1, 1, 1, 2, 2, 2, 4, \ldots\}
\]

\[
\alpha_n(\mathbb{Z}, 5) = \{0, 0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots\}
\]

\[6! = 2^4 \cdot 3^2 \cdot 5\]
Generalized Factorial on $S$

Let $S = \{0, 1, 8, 27, 64, 125\}$, we will calculate $3!_S$, the factorial of an element with index 3 in our $p$-ordering, in our case this was 8:

\[
\begin{align*}
\alpha_n(\mathbb{Z}, 2) &= \{0, 0, 1, 3, 3, 9\} \\
\alpha_n(\mathbb{Z}, 3) &= \{0, 0, 0, 2, 2, 3\} \\
\alpha_n(\mathbb{Z}, 5) &= \{0, 0, 0, 0, 0, 3\} \\
\alpha_n(\mathbb{Z}, 7) &= \{0, 0, 0, 1, 2, 2\}
\end{align*}
\]

\[3!_S = 2^3 \cdot 3^2 \cdot 5^0 \cdot 7^1\]
Generalized Factorial on $S$

Let $S = \{0, 1, 8, 27, 64, 125\}$, we will calculate $3!_S$, the factorial of an element with index 3 in our $p$-ordering, in our case this was 8:

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\alpha_n(\mathbb{Z}, 7) = \{0, 0, 0, 1, 2, 2\}
$$

$$
3!_S = 2^3 \cdot 3^2 \cdot 5^0 \cdot 7^1
$$
Basis for IVPs on a subset of $\mathbb{Z}$

**Theorem (Bhargava)**

A polynomial is integer valued on a subset $S$ of $\mathbb{Z}$ if and only if it can be written as a $\mathbb{Z}$-linear combination of the polynomials

$$
\frac{B_{k,S}}{k!_S} = \frac{(x - a_{0,k})(x - a_{1,k}) \cdots (x - a_{k-1,k})}{k!_S},
$$

for $k = 0, 1, 2, \ldots$ where the $B_{k,S}(x)$ are the polynomials defined by

$$(x - a_{0,k})(x - a_{1,k}) \cdots (x - a_{k-1,k}),$$

where $\{a_{i,k}\}_{i=0}^{\infty}$ is a sequence in $\mathbb{Z}$ that, for each prime $p$ dividing $k!_S$, is term-wise congruent modulo $\alpha_k(S, p)$ to some $p$-ordering of $S$. 

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**Definition (Chabert)**

The characteristic ideal of index $n$ of $\mathbb{Z}$ is the set $\mathcal{J}_n(\mathbb{Z})$ formed by 0 and the leading coefficients of the polynomials in $\text{Int}(S, \mathbb{Z})$ of degree $n$.

For example, when $S = \mathbb{Z}$, $\mathcal{J}_n(\mathbb{Z}) = \frac{1}{n!} \mathbb{Z}$.

**Definition (Chabert)**

The characteristic sequence of $S$ with respect to a fixed prime $p$ is the sequence of negatives of the $p$-adic valuations of these ideals, denoted by $\alpha_n(S, p)$. 
For $S$ a subset of $\mathbb{Z}$ and $p$ a fixed prime, the valuative capacity of $S$ with respect to the prime $p$ is the following limit:

$$L_{S, p} = \lim_{n \to \infty} \frac{\alpha_n(S, p)}{n}.$$ 

The positive integers in increasing order are a $p$-ordering of $\mathbb{Z}$ and we have that $\alpha_n(\mathbb{Z}, p) = \nu_p(n!)$. By Legendre's formula $\nu_p(n!) = \frac{n - \sum n_i}{p - 1}$, we can compute

$$L_{\mathbb{Z}, p} = \lim_{n \to \infty} \frac{\alpha_n(\mathbb{Z}, p)}{n} = \frac{1}{p - 1}.$$
Let \( S \) be the set of perfect squares in \( \mathbb{Z} \) and

\[
E = S + S \quad F = S + S + S
\]

**Theorem (Fares, Johnson)**

\[
L_{E,p} = \begin{cases} 
\frac{1}{p-1} & \text{if } p \equiv 1 \pmod{4} \\
-1 + \sqrt{1 + \frac{2p}{(p-1)^2}} & \text{if } p \equiv 3 \pmod{4} \\
\frac{-1 + \sqrt{13}}{2} & \text{if } p = 2
\end{cases}
\]

\[
L_{F,p} = \begin{cases} 
\frac{1}{p-1} & \text{if } p > 2 \\
\frac{-25 + 3\sqrt{705}}{52} & \text{if } p = 2
\end{cases}
\]
Sums of $\ell$ Elements to the Power of $d$

For $D = \{x^d \mid x \in \mathbb{Z}\}$ and we let $\ell D = D + \cdots + D$, for $\ell$ terms in the sum.

**Theorem (B.)**

Suppose $p$ is a prime and $d = p^j d'$ a positive integer not equal to $4$, where $p \nmid d'$ and let $e = 2j + 1$.

Then, $L_{\ell D,p}$ is an algebraic number of degree at most 2.

When 0 can be written non-trivially as a sum of $\ell$ elements to the power of $d \pmod{p^e}$, $L_{\ell D,p}$ is a rational number.

**Corollary (B.)**

For a fixed $\ell$, if $d$ is odd and $p$ is a prime, then $L_{\ell D,p} \in \mathbb{Q}$. 
Timeline of IVPs

- Pólya in 1915
- Cahen and Chabert wrote a textbook on the subject, mid 90’s
- Bhargava, $p$-sequences, late 90’s
- Chabert, valuative capacity 2001
- Chabert, survey article 2014
Generalization

• Recall that the official definition is $\text{Int}(S, D)$.

• IVPs over quaternions. [Werner] [Johnson and Pavlovski]

• IVPs over matrix rings. [Evrard, Fares and Johnson] [Frisch] [Werner]

• Multivariable case. [Bhargava] [Evrard]

• Homogeneous 2-variable case. [Johnson and Patterson]
Let \( n \) be a positive integer, let \( S \) be a subset of \( \mathbb{Z}^n \) and let \( m_0, m_1, \ldots \) be an ordering of the monomials of \( \mathbb{Z}^n[X] \), and consider the \( \mathbb{Z} \)-algebra

\[
\text{Int}(S) = \{ f(X_1, \ldots, X_n) \in \mathbb{Q}[X_1, \ldots, X_n] \mid f(S) \subseteq \mathbb{Z} \}.
\]

Using matrices, \( p \)-orderings and \( p \)-sequences can be generalized to the multivariable case.
Proposition

The polynomials \( \left\{ \left( \frac{x_1}{r_1} \right) \left( \frac{x_2}{r_2} \right) \cdots \left( \frac{x_n}{r_n} \right) \mid r_1, \ldots, r_n \in \mathbb{Z}, \ r_1, \ldots, r_n \geq 0 \right\} \)
form a basis of the \( \mathbb{Z} \)-module of \( \text{Int}(\mathbb{Z}^n) \).

In the three variable case, the elements of degree 3 are

\[
\begin{align*}
(x_1^3), \quad (y_1^3), \quad (z_1^3), \quad (x_1^2)(y_1^1), \quad (x_1^2)(z_1^1), \quad (x_1^1)(y_1^2), \quad (y_1^2)(z_1^1), \\
(x_1^1)(z_1^2), \quad (y_1^1)(z_1^2), \quad (x_1^1)(y_1^1)(z_1^1)
\end{align*}
\]
Homogeneous Polynomials

A homogeneous polynomial is one of the form

\[ f(x_1, \ldots, x_n) = \sum_{i_1 + i_2 + \cdots + i_n = m} c_1 x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}. \]

For degree \( m \) they have the property that for a constant \( h \):

\[ f(hx_1, \ldots, hx_n) = h^m f(x_1, \ldots, x_n) \]
Homogeneous as a Product of Linears

- When written as a product of linear factors each term must contain a variable, we can't have \((x - 2)\) we would need \((x - 2y)\).

- Start by focusing on how big of a power of \(p = 2\) we can get. How can we construct 2-variable IVPs?

- In degree 1 and 2 we need integer coefficients: \(x, y, xy\).

- Can we obtain a denominator in degree 3?
Homogeneous as a Product of Linears

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- In degree 1 and 2 we need integer coefficients: \(x, y, xy\).

- Can we obtain a denominator in degree 3?

- Yes! \(\frac{xy(x-y)}{2}\)
Basis for Degree 3, 3 Variables

In degree $m$ we will have $\frac{(m+1)(m+2)}{2}$ basis elements.

$$\{ xyz, \frac{xy(x - y)}{2}, \frac{xz(x - z)}{2}, \frac{yz(y - z)}{2}, x^2(x - y), x^2(x - z),$$
$$y^2(y - x), y^2(y - z), z^2(z - x), z^3 \}.$$
Another Example

- For example when $m = 6$, we have 7 elements with a $2^0$, 14 with a $2^1$, 4 with a $2^2$ and 3 with a $2^3$ in their denominators.

- The polynomials with a $2^3$ in their denominator are:

$$f = \frac{1}{4}x^5y + \frac{7}{8}x^4y^2 + \frac{1}{8}x^2y^4 + \frac{3}{4}xy^5$$

$$g = \frac{3}{4}x^5z + \frac{1}{8}x^4z^2 + \frac{7}{8}x^2z^4 + \frac{1}{4}xz^5$$

$$h = \frac{1}{4}x^5y + \frac{1}{2}x^5z + \frac{3}{8}x^4y^2 + \frac{3}{4}x^4z^2 + \frac{5}{8}x^2y^4 + \frac{1}{4}x^2z^4$$

$$+ \frac{3}{4}xy^5 + \frac{1}{2}xz^5 + \frac{3}{4}y^5z + \frac{1}{8}y^4z^2 + \frac{7}{8}y^2z^4 + \frac{1}{4}yz^5$$
Counting the Basis Elements

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Thank you

Thanks for listening to this presentation.