Irreducibility and Roots of a Class of Polynomials
(Number Theory Seminar)

Abdullah Al-Shaghay

Dalhousie University

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Overview

1. Introduction
2. What About a Gap?
3. Pictures
4. Future Directions
Overview

1 Introduction

2 What About a Gap?

3 Pictures

4 Future Directions
In a 2012 paper, J. Harrington investigated the factorization properties of polynomials of the form

\[ f(x) = x^n + cx^{n-1} + cx^{n-2} + \ldots + cx + c \in \mathbb{Z}[x]. \]

In particular, it was asked:

- For what positive integers \( n \) and nonzero integers \( c \) is \( f(x) \) irreducible?
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- For what positive integers \( n \) and nonzero integers \( c \) is \( f(x) \) irreducible?
- If \( f(x) \) is reducible, then how does it factor?
Certain cases of this problem have already been answered:

- If there exists a prime $p$ such that $p|\mid c$, then $f(x)$ is irreducible for all values $n$ by the Eisenstein criterion.
Certain cases of this problem have already been answered:

- If there exists a prime $p$ such that $p || c$, then $f(x)$ is irreducible for all values $n$ by the Eisenstein criterion.

- If $c = 1$, $f(x)$ is irreducible if and only if $n = p - 1$ for an odd prime $p$. 
Main Result

**Theorem**

Let $n$ and $c$ be positive integers with $c \geq 2$. Then the polynomials

\[
\begin{align*}
    f(x) &= x^n + \sum_{j=0}^{n-1} cx^j, \\
    g(x) &= x^n + \sum_{j=0}^{n-1} (-1)^{n-j} cx^j, \\
    h(x) &= x^n - \sum_{j=0}^{n-1} cx^j, \\
    k(x) &= x^n - \sum_{j=0}^{n-1} (-1)^{n-j} cx^j,
\end{align*}
\]

are irreducible in $\mathbb{Z}[x]$ with the exceptions of:

\[
\begin{align*}
    f(x) &= x^2 + 4x + 4 = (x + 2)^2 \quad \text{and} \quad g(x) = x^2 - 4x + 4 = (x - 2)^2.
\end{align*}
\]
Theorem

Let \( n, c, \) and \( d \) be positive integers with \( n \geq 3, d \neq c, d \leq 2(c - 1), \) and \((n, c) \neq (3, 3)\). If the trinomial \( f(x) = x^n \pm cx^{n-1} \pm d \) is reducible in \( \mathbb{Z}[x] \), then \( f(x) = (x \pm 1)g(x) \) for some irreducible \( g(x) \in \mathbb{Z}[x] \).

Theorem (Rouché’s Theorem)

Given two functions \( f \) and \( g \) analytic inside some region \( K \) with a closed contour as \( \partial K \), if \( |g(z)| < |f(z)| \) on \( \partial K \), then \( f \) and \( f + g \) have the same number of zeros inside \( K \) (counting multiplicity).
Lemma

Let \( f(x) \in \mathbb{Z}[x] \) be a monic polynomial with \( f(0) \neq 0 \). If \( f(x) \) has only one root (counting multiplicity) in \( \mathbb{C}\setminus\mathbb{U} \), then \( f(x) \) is irreducible in \( \mathbb{Z}[x] \).

Lemma

Let \( f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 \in \mathbb{Q}[x] \) with \( n \geq 1 \) and \( a_0 \neq 0 \). If \( |a_{n-1}| > 1 + \sum_{j=0}^{n-2} |a_j| \), then \( f(x) \) has exactly one root (counting multiplicity) in \( \mathbb{C}\setminus\mathbb{U} \).
Lemma

Let $n, m, c, \text{ and } d$ be positive integers with $n > m$ and $d \geq c + 1$. Then the trinomial $f(x) = x^n \pm cx^m \pm d$ has no roots in $\mathcal{U}$. Furthermore, if $d > c + 1$, then $f(x)$ has no roots in $\overline{\mathcal{U}}$.

Lemma

Let $n, c \text{ and } d$ be positive integers with $n \geq 2$ and $d < (c - 1)^{n-1}$. Then the trinomial $f(x) = x^n \pm cx^{n-1} \pm d$ has a root $\alpha \in \mathbb{R}$ with $|\alpha| > c - 1$. 
Lemma

Let $K$ be a positive integer and let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with no roots in the set $\{z \in \mathbb{C} : |z| \leq K\}$. If $f(x)$ has a root $\alpha$ with $|\alpha| > \frac{|f(0)|}{K+1}$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Cases are then considered:

- $d < c - 1$ and use first 2 lemmas.
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- $c + 1 < d \leq 2(c - 1)$ and use the last 3 lemmas.
Lemma

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Cases are then considered:

- $d < c - 1$ and use first 2 lemmas.
- $c + 1 < d \leq 2(c - 1)$ and use the last 3 lemmas.
- $d = c \pm 1$ are dealt with separately.
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For what positive integers $n$ and $c$ is $f(x)$ irreducible?
New Polynomials, Old Questions

\[ f(x) = x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c \in \mathbb{Z}[x]. \]

- For what positive integers \( n \) and \( c \) is \( f(x) \) irreducible?
- If \( f(x) \) is reducible, then how does it factor?
Conjecture

Let $n$, $c$, and $a$ be positive integers. If the quadranomial $h(x) = x^{n+1} - x^n + cx^{n-a} - c$ is reducible in $\mathbb{Z}[x]$, then $h(x) = (x - 1)g(x)$ for some irreducible $g(x) \in \mathbb{Z}[x]$. 
Irreducibility

Theorem

Let $n, a,$ and $c$ be positive integers with $c \geq 2$. Then the polynomials

$f(x) = x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c$ are irreducible in $\mathbb{Z}[x]$. 
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Let us take a look at the roots of

\[ x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c = f_{n,a,c}(x) \]

for different values of \( n, a, \) and \( c. \)
\( (n, a, c) = (10, 3, 8) \)
\[(n, a, c) = (15, 3, 8)\]
\[(n, a, c) = (20, 3, 8)\]
\((n, a, c) = (50, 3, 8)\)
\[(n, a, c) = (50, 3, 8) \text{ vs. } (n, a, c) = (50, 3, 18)\]
$(n, 3, 8) \quad 1 \leq n \leq 50$
\((n, a, c) = (50, 3, 8)\) vs. \(g(x) = (x^4 - x^3 + 8) \cdot (\sum_{j=0}^{46} x^j)\)
\((n, a, c) = (50, 4, 8)\) vs. \(h(x) = (x^5 - x^4 + 8) \cdot (\sum_{j=0}^{45} x^j)\)
\((n, a, c) = (50, 5, 8)\) vs. \(r(x) = (x^6 - x^5 + 8) \cdot \left(\sum_{j=0}^{44} x^j\right)\)
\[(n, a, c) = (50, 6, 8) \text{ vs. } s(x) = (x^7 - x^6 + 8) \cdot \left(\sum_{j=0}^{43} x^j\right)\]
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What Next?

- Negative values of $c$. 

Alternating coefficients; introduce a product of $(−1)^{j−1}$. I'm curious to know how certain parameters affect the "rate of convergence" in comparison to one-another.
What Next?

- Negative values of $c$.
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Negative values of $c$.

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I’m curious to know how certain parameters affect the “rate of convergence” in comparison to one-another.
Negative values of $c$

Things are looking irreducible except for certain situations ($x^2 - a^2$ and $x^3 - a^3$).
Negative values of $c$

- Things are looking irreducible except for certain situations ($x^2 - a^2$ and $x^3 - a^3$).
- Roots have the same behaviour!
Alternating Coefficients

- Things are looking irreducible ... no exceptions this time.
Things are looking irreducible ... no exceptions this time.
Roots have “essentially” the same behaviour!
Thank you very much for your time and patience! Please feel free to ask any questions and I will do my best to answer them.