A Linear Algebra Problem Related to Legendre Polynomials

Scott Cameron

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Find \( f(x) \) such that

\[
g\left(\frac{1}{2}\right) = \int_{0}^{1} f(x)g(x) \, dx
\]

where \( f(x) \) and \( g(x) \) are polynomials of degree \( \leq 2 \).
To solve let

\[ f(x) = ax^2 + bx + c \]

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Now we simply multiply these and integrate from 0 to 1. The result of this must be equal to \( g(\frac{1}{2}) \).
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Now we simply multiply these and integrate from 0 to 1. The result of this must be equal to \( g(\frac{1}{2}) \).

Then we just allow the coefficients of \( \alpha, \beta, \text{and} \gamma \) to be equal and solve a system of equations.
The leads to the solution

\[ f(x) = -15x^2 + 15x - \frac{3}{2} \]
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After finding this solution I wanted to have some fun and see how the answer changed if I made a small change to the original problem.
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Find $f(x)$ such that

$$g\left(\frac{1}{2}\right) = \int_0^1 f(x)g(x)\,dx$$

where $g(x)$ and $f(x)$ are polynomials of degree $\leq 3$. 
This problem is of course solved in the exact same way as the previous, however I did not want to solve the system of equations by hand, and so I taught Maple how to solve the problem and asked for some help.

To test if Maple understood, I asked what if $\deg \leq 2$?

Maple responds $f(x) = -15x^2 + 15x - \frac{3}{2}$.

Good.

So then what if $\deg \leq 3$?

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Same thing again.
I continued for a few more degrees, and the pattern continued as well.

I thought maybe it is somehow possible that I made a mistake that only appears for odd degrees, so I checked if the answer held up with the original problem.

I asked Maple for a random polynomial of degree $\leq 3$, and integrated it against $f(x) = -15x^2 + 15x - 32$.

The answer was the random polynomial at $x = \frac{1}{2}$. Perhaps no mistake was made.

But why is this happening?

To answer this we will state the question again, but in more general terms.
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Find $f_n(x)$ such that

$$g(c) = \int_{0}^{1} f_n(x) g(x) \, dx$$

where $g(x)$ and $f_n(x)$ are polynomials of degree $\leq n$. 

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Generalizing the Problem

Find $f_n(x)$ such that

$$g(c) = \int_0^1 f_n(x)g(x)\,dx$$

where $g(x)$ and $f_n(x)$ are polynomials of degree $\leq n$.

Now let us write our question in terms of the following proposition.
Proposition

Let $f_n(x)$ be as previously defined. Then when $c = \frac{1}{2}$ we have $f_{2m+1}(x) = f_{2m}(x)$ for $m \in \mathbb{N}$.
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Let \( f_n(x) \) be as previously defined. Then when \( c = \frac{1}{2} \) we have 
\[ f_{2m+1}(x) = f_{2m}(x) \text{ for } m \in \mathbb{N}. \]

So then let 
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f_n(x) = \sum_{k=0}^{n} a_k x^k
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Now write this as $Ha = c$. The choice of $H$ is because matrices of this form ($H_{ij} = \frac{1}{i+j-1}$) are known as Hilbert matrices.

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$$(H_{i,j})^{-1} = (-1)^{i+j-1}(i+j-1) \binom{n+i}{i+j-1} \binom{n+j}{i+j-1} \binom{i+j-2}{i-1}^2.$$
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Luckily Hilbert matrices have a known formula for the entries of their inverse.

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Before continuing we need a definition.
Definition

Let $h_{n,i}(c)$ be the polynomial created by taking the dot product of the $i^{th}$ row of $H^{-1}$ and $c$.

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These polynomials determine the coefficients of $f_n(x)$. That is,

$$h_{n,i}(c) = a_{i-1}, \text{ or } f_n(x) = \sum_{k=1}^{n+1} h_{n,k}(c)x^{k-1}$$
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Let’s do an example to clarify.
Example: If $n = 2$, then

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$
Example: If $n = 2$, then

\[
H = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 3 \\
1 & 4 & 4 \\
1 & 5 & 5 \\
\end{bmatrix}.
\]

Using our formula,

\[
H^{-1} = \begin{bmatrix}
9 & -36 & 30 \\
-36 & 192 & -180 \\
30 & -180 & 180 \\
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So we have \( n = 2 \) and can see that \( 1 \leq i \leq 3 \). Thus

\[
h_{2,1}(c) = 9 - 36c + 30c^2,
\]
\[
h_{2,2}(c) = -36 + 192c - 180c^2,
\]
\[
h_{2,3}(c) = 30 - 180c + 180c^2,
\]
\[
f_n(x) = h_{2,1}(c) + h_{2,2}(c)x + h_{2,3}(c)x^2
\]
Now back to our problem.
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If \( n \) is odd, and \( f_n(x) = f_{n-1}(x) \) when \( c = \frac{1}{2} \), then the degree of \( f_n(x) \) is \( n - 1 \). This means that \( a_n = 0 \) in \( f_n(x) \).
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This would correspond to the polynomial formed by the bottom row of \( H^{-1} \) having a root at \( c = \frac{1}{2} \). Using our definition, this can be written as \( h_{n,n+1}(c) \) having a root at \( c = \frac{1}{2} \).
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Using the formula for \( H^{-1} \) we can write \( h_{n,n+1}(c) \) as

\[
h_{n,n+1}(c) = \sum_{j=1}^{n+1} (-1)^{n+j+1}(n+j) \binom{2n+1}{n+j} \binom{n+j}{n+j} \binom{n+j-1}{n}^2 c^{j-1}
\]
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\[ h_{n,n+1}(c) = (-1)^n (2n + 1) \binom{2n + 1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} c^k. \]
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This form is exactly what we need. The sum is our polynomial, and then we have a scaling factor outside.
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This form is exactly what we need. The sum is our polynomial, and then we have a scaling factor outside.

All we need now is a definition to solve our problem.
The shifted Legendre polynomials, denoted $\tilde{P}_n(x)$, are given by

$$\tilde{P}_n(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k.$$
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Looking back at our expression for $h_{n,n+1}(c)$,

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we can now see that $h_{n,n+1}(c)$ is just a multiple of $\tilde{P}_n(c)$. 
The shifted Legendre polynomials are so named because they are, unsurprisingly, Legendre polynomials which have been shifted.

\[ P_n(2x - 1) = \tilde{P}_n(x) \]

The Legendre polynomials are known to have \( x = 0 \) as a root when their degree is odd. Therefore, the shifted Legendre polynomials must have a root at \( x = \frac{1}{2} \) when their degree is odd.
The shifted Legendre polynomials are so named because they are, unsurprisingly, Legendre polynomials which have been shifted.

The shift is given by sending $x$ to $2x - 1$. That is, if we denote the Legendre polynomials by $P_n(x)$, then $P_n(2x - 1) = \tilde{P}_n(x)$. 
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The shifted Legendre polynomials have a root at $\frac{1}{2}$ when their degree is odd.

Therefore, if $n$ is odd and $c = \frac{1}{2}$, then $a_n = 0$. This ends up forcing $f_n(x) = f_{n-1}(x)$, thus answering our question.
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That is, how does \( h_n, 1 \) change as we change \( n \)? etc.
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However, when I found this solution, it gave me another question.

If $h_{n,n+1}(c)$ is always just a multiple of a shifted Legendre polynomial, what do other rows correspond to?
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That is, how does $h_{n,1}(c)$ change as we change $n$? $h_{n,2}(c)$? etc.
Another Approach to the Problem and the Other Rows of $H^{-1}$

Until now, I have been using just basic calculus and matrix operations to answer these questions. There is however a better way.
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Theorem

(Riesz Representation Theorem) *If we have some finite dimensional vector space, $V$, and some linear functional $\phi$ on $V$, then there is a unique vector $u \in V$ such that*

$$\phi(v) = \langle v, u \rangle$$

*for all $v \in V$.*
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We can interpret our problem in terms of this theorem. The integral is an inner product, $g(x)$ corresponds to $v$, $f_n(x)$ corresponds to $u$, and evaluation at $c$ is a linear functional.
This theorem has the consequence of allowing us to write

\[ f_n(x) = \sum_{k=0}^{n} \frac{\tilde{P}_k(c) \tilde{P}_k(x)}{\int_{0}^{1} \tilde{P}_k(x)^2 \, dx} \]

\[ = \sum_{k=0}^{n} (2k + 1) \tilde{P}_k(c) \tilde{P}_k(x) \]
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It should be noted that this expression for \( f_n(x) \) shows us that it is actually a familiar concept in the study of orthogonal polynomials.

In this form \( f_n(x) \) would be called the kernel of the shifted Legendre polynomials. Therefore what I am studying can be interpreted as looking at how the coefficients of this kernel change with \( n \), and with \( c \).
Moving back to \( h_{n,i}(c) \), we can use the previous expression of \( f_n(x) \) and the fact that

\[
f_n(x) = \sum_{k=1}^{n+1} h_{n,k}(c)x^{k-1}
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$$f_n(x) = \sum_{k=1}^{n+1} h_{n,k}(c)x^{k-1}$$

To find that

$$h_{n,i}(c) = \sum_{k=i-1}^{n} (-1)^{k+i-1} \binom{k}{i-1} \binom{k+i-1}{i-1} (2k + 1) \tilde{P}_k(c)$$
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Using this equation, I wanted to find a generating function for these polynomials.
First let us focus on $i = 1$. 

If $i = 1$ we have $h_{n,1}(c) = \sum_{k=0}^{n}(-1)^k(2k+1)\tilde{P}_k(c)$. 

Now we need to make use of a relationship between the shifted Legendre polynomials. 

$$(n+1)\tilde{P}_{n+1}(x) = (2n+1)(2x-1)\tilde{P}_n(x) - n\tilde{P}_{n-1}(x)$$ 

Combining the two expressions, it can be shown that $h_{n,1}(c) = (-1)^n\frac{(n+1)}{2c}(\tilde{P}_n(c) + \tilde{P}_{n+1}(c))$. 

Using this we can find a generating function for $h_{n,1}(c)$. 

Scott Cameron  
A Linear Algebra Problem Related to Legendre Polynomials
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Rearranging, multiplying by $x^{n+1}$, and summing over $n$ yields

\[
\sum_{n=0}^{\infty} \frac{h_{n,1}(c)x^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{2c} \left( \tilde{P}_n(c)(-x)^{n+1} + \tilde{P}_{n+1}(c)(-x)^{n+1} \right)
\]

\[
= -\frac{1}{2c} \left( -x \sum_{n=0}^{\infty} \tilde{P}_n(c)(-x)^n + \sum_{n=0}^{\infty} \tilde{P}_n(c)(-x)^n - \tilde{P}_0(c) \right)
\]

\[
= -\frac{1}{2c} \left( (1 - x) \sum_{n=0}^{\infty} \tilde{P}_n(c)x^n - 1 \right)
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Now we take the derivative with respect to $x$ of both sides which gives us the generating function.
Let $\mathcal{H}_1(x)$ be the generating function for $h_{n,1}(c)$. 
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Then we have from the previous slide

$$\mathcal{H}_1(x) = -\frac{1}{2c} \frac{d}{dx} \left( \frac{1 - x}{\sqrt{1 + 2(2c - 1)x + x^2}} - 1 \right)$$

$$= \frac{1 + x}{(1 + 2(2c - 1)x + x^2)^{3/2}}$$
Now using our expression for $H_1(x)$, along with

$$h_{n,i}(c) = \sum_{k=i-1}^{n} (-1)^{k+i-1} \binom{k}{i-1} \binom{k+i-1}{i-1} (2k+1) \tilde{P}_k(c)$$

we can find an expression for the generating function of any $h_{n,i}(c)$, denoted $H_i(x)$. 
Sparing the details of the calculation, as it is more complicated but similar to the derivation of $H_1(x)$, the final result is given by

$$H_i(x) = \frac{(-x)^{i-1}}{(1-x)((i-1)!)^2} \frac{d^{2i-2}}{dx^{2i-2}} \left( \frac{x^{i-1}(1-x)(1+x)}{(1 + 2(2c - 1)x + x^2)^{3/2}} \right)$$
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If we let $j = i - 1$ then this takes on a nicer form of

$$\mathcal{H}_{i+1}(x) = \frac{(-x)^{j}}{(1-x)(j!)^2} \frac{d^{2j}}{dx^{2j}} \left( \frac{x^{j}(1-x)(1+x)}{(1 + 2(2c - 1)x + x^2)^{\frac{3}{2}}} \right)$$
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So we have accomplished our goal of finding the generating function for the polynomials $h_{n,i}(c)$. 
Here are the first couple generating functions.

\[ H_1(x) = 1 + x (1 + 2 (c - 1)x + x^2) \]

\[ H_2(x) = 12x (1 + x) ((c - 1/2)x^2 - (c^2 - c - 1/2)x + c - 1/2) (1 + 2 (2c - 1)x + x^2) \]

\[ H_3(x) \] is a bit long so I will break it up a bit. The denominator is the same as the others but with exponent 11/2. There is also the term \( 180x^2 (1 + x) \) which is also similar to the others. The part I want to show however, is the polynomial in the numerator.
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$$\left((c - \frac{1}{2})x^2 - (c^2 - c - 1)x + c - \frac{1}{2}\right).$$

And for $H_3(x)$

$$\left(c^2 - c + \frac{1}{6}\right)x^4 + \left(-\frac{4}{3}c^3 + 2c^2 + \frac{2}{3}c - \frac{2}{3}\right)x^3$$

$$-\frac{1}{3}(c^2 - c + 3)(c^2 - c - 1)x^2$$

$$+ \left(-\frac{4}{3}c^3 + 2c^2 + \frac{2}{3}c - \frac{2}{3}\right)x + \left(c^2 - c + \frac{1}{6}\right).$$
Now I will quickly state some other results and properties of these polynomials.
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\int_0^1 h_{n,i}(c) c^n dc = \begin{cases} 
1 & \text{if } i = n + 1 \\
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$$\int_0^1 h_{n,i}(c)c^n dc = \begin{cases} 
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In other words, integrating $h_{n,i}(c)$ against another polynomial in $c$ of degree at least $n$, will be equal to the coefficient of $c^{i-1}$ of the polynomial.
Another representation for these polynomials is

$$h_{n,i}(c) = (-1)^i i \binom{n+i}{i} \binom{n+1}{i} _3F_2(-n, n+2, i; 1, i+1; c)$$
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This just gives us another representation of the polynomials which can be investigated. Given this representation I was also able to prove the following identity

\[
\int_0^1 _3F_2(-n, n + 2, i; 1, i + 1; c) \, _3F_2(-m, m + 2, i; 1, i + 1; c) \, dc = \frac{i^2 \Gamma(n + i + 1) \Gamma(m + 2 - i)}{(2i - 1)(n + 1)(m + 1) \Gamma(n + 2 - i) \Gamma(m + i + 1)}
\]
Next, I would like to show you some images that arise from the hypergeometric representation of the polynomials.
A Linear Algebra Problem Related to Legendre Polynomials
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Thanks for listening.
Special thanks to Dalhousie University and to my advisor, Karl Dilcher