$p$-adic valuations of certain colored partition functions

Maciej Ulas

Institute of Mathematics, Jagiellonian University, Kraków, Poland

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The general question
Short plan of the presentation

- The general question
- The Prouhet-Thue-Morse sequence and the binary partition function
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- A general result
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- 2-adic valuations for all powers
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- The Prouhet-Thue-Morse sequence and the binary partition function
- A general result
- 2-adic valuations for all powers
- Some results for $p$-ary colored partitions
In the sequel we will use the following notation:

- \( \mathbb{N} \) denote the set of non-negative integers,
- \( \mathbb{N}_+ \) - the set of positive integers,
- \( \mathbb{P} \) - the set of prime numbers,
- \( \mathbb{N}_{\geq k} \) - the set \( \{ n \in \mathbb{N} : n \geq k \} \).
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If \( p \in \mathbb{P} \) and \( n \in \mathbb{Z} \) we define the \( p \)-adic valuation of \( n \) as:

\[
\nu_p(n) := \max\{ k \in \mathbb{N} : p^k \mid n \}.
\]

We also adopt the standard convention that \( \nu_p(0) = +\infty \).
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From the definition we easily deduce that for each \( n_1, n_2 \in \mathbb{Z} \) the following properties hold:

\[
\nu_p(n_1 n_2) = \nu_p(n_1) + \nu_p(n_2) \quad \text{and} \quad \nu_p(n_1 + n_2) \geq \min\{ \nu_p(n_1), \nu_p(n_2) \}.
\]

If \( \nu_p(n_1) \neq \nu_p(n_2) \) then the inequality can be replaced by the equality.
Let

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[[x]] \]

and

\[ g(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{Z}[[x]] \]

be a formal power series with integer coefficients and \( M \in \mathbb{N}_{\geq 2} \) be given. We say that \( f, g \) are congruent modulo \( M \) if and only if for all \( n \) the coefficients of \( x^n \) in both series are congruent modulo \( M \).
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In other words
\[ f \equiv g \pmod{M} \iff \forall n \in \mathbb{N} : a_n \equiv b_n \pmod{M}. \]
One can prove that for any given \( f, F, g, G \in \mathbb{Z}[[x]] \) satisfying
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    f \equiv g \pmod{M} \quad \text{and} \quad F \equiv G \pmod{M}
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Moreover, if \( f(0), g(0) \in \{-1, 1\} \) then the series \( 1/f, 1/g \) have integer coefficients and we also have
\[
\frac{1}{f} \equiv \frac{1}{g} \pmod{M}.
\]
In consequence, in this case we have
\[
f^k \equiv g^k \pmod{M}
\]
for any \( k \in \mathbb{Z} \).
We formulate the following general

**Question 1**

Let $f(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n \in \mathbb{Z}[[x]]$ with $\varepsilon_0 \in \{-1, 1\}$ and take $m \in \mathbb{N}_+$. What can be said about the sequences $(\nu_p(a_m(n)))_{n \in \mathbb{N}}, (\nu_p(b_m(n)))_{n \in \mathbb{N}}$, where

\[
 f(x)^m = \left( \sum_{n=0}^{\infty} \varepsilon_n x^n \right)^m = \sum_{n=0}^{\infty} a_m(n)x^n, \\
 \frac{1}{f(x)^m} = \left( \sum_{n=0}^{\infty} \frac{1}{\varepsilon_n x^n} \right)^m = \sum_{n=0}^{\infty} b_m(n)x^n,
\]

i.e., $a_m(n)$ ($b_m(n)$) is the $n$-th coefficient in the power series expansion of the series $f^m(x)$ ($1/f(x)^m$ respectively)?
It is clear that in its full generality, the Question 1 is too difficult and we cannot expect that the sequences \((\nu_p(a_m(n)))_{n\in\mathbb{N}}\) and \((\nu_p(b_m(n)))_{n\in\mathbb{N}}\) can be given in closed form or even that a reasonable description can be obtained. Indeed, in order to give an example let us consider the formal power series

\[
f(x) = \prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n (x \frac{n(3n-1)}{2} + x \frac{n(3n+1)}{2}).
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The second equality is well know theorem: the Euler pentagonal number theorem.
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In particular $a(n) \in \{-1, 0, 1\}$ and thus for any given $p \in \mathbb{P}$ we have $\nu_p(a(n)) = 0$ in case when $n$ is of the form $n = \frac{m(3m\pm 1)}{2}$ for some $m \in \mathbb{N}_+$, and $\nu_p(a(n)) = \infty$ in the remaining cases.
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However, the characterization of the 2-adic behaviour of the sequence \( (p(n))_{n \in \mathbb{N}} \) given by

\[
\frac{1}{f(x)} = \prod_{n=1}^{\infty} \frac{1}{1 - x^n} = 1 + \sum_{n=1}^{\infty} p(n)x^n
\]

is unknown. Let us note that the number \( p(n) \) counts the integer partitions of \( n \), i.e., the number of non-negative integer solutions of the equation \( \sum_{i=1}^{n} x_i = n \). In fact, even the proof that \( \nu_2(p(n)) > 0 \) infinitely often is quite difficult.
Let \( n \in \mathbb{N} \) and \( n = \sum_{i=0}^{k} \varepsilon_i 2^i \) be the unique expansion of \( n \) in base 2 and define the sum of digits function

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s_2(n) = \sum_{i=0}^{k} \varepsilon_i.
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Next, we define the Prouhet-Thue-Morse sequence \( t = (t_n)_{n \in \mathbb{N}} \) (on the alphabet \( \{-1, +1\} \)) in the following way

\[
t_n = (-1)^{s_2(n)},
\]

i.e., \( t_n = 1 \) if the number of 1’s in the binary expansion of \( n \) is even and \( t_n = -1 \) in the opposite case. We will call the sequence \( t \) as the PTM sequence in the sequel.
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From the relations

\[
s_2(0) = 0, \quad s_2(2n) = s_2(n), \quad s_2(2n+1) = s_2(n) + 1
\]

we deduce the recurrence relations for the PTM sequence: \( t_0 = 1 \) and

\[
t_{2n} = t_n, \quad t_{2n+1} = -t_n.
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Let

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In consequence we easily deduce the representation of \( T \) in the infinite product shape

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Let us also note that the (multiplicative) inverse of the series \( T \), i.e.,

\[ B(x) = \frac{1}{T(x)} = \prod_{n=0}^{\infty} \frac{1}{1 - x^{2^n}} = \sum_{n=0}^{\infty} b_n x^n \]

is an interesting object.
Indeed, for $n \in \mathbb{N}$, the number $b_n$ counts the number of binary partitions of $n$. The binary partition is the representation of the integer $n$ in the form

$$n = \sum_{i=0}^{n} u_i 2^i,$$

where $u_i \in \mathbb{N}$ for $i = 0, \ldots, n$. 
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The sequence $(b_n)_{n \in \mathbb{N}}$ was introduced by Euler. However, it seems that the first nontrivial result concerning its arithmetic properties was obtained by Churchhouse. He proved that $\nu_2(b_n) \in \{1, 2\}$ for $n \geq 2$. 
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More precisely, $b_0 = 1$, $b_1 = 1$ and for $n \geq 2$ we have $\nu_2(b_n) = 2$ if and only if $n$ or $n - 1$ can be written in the form $4^r(2u + 1)$ for some $r \in \mathbb{N}_+$ and $u \in \mathbb{N}$. In the remaining cases we have $\nu_2(b_n) = 1$. 
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We can compactly write

$$\nu_2(b_n) = \begin{cases} \frac{1}{2} |t_n - 2t_{n-1} + t_{n-2}|, & \text{if } n \geq 2 \\ 0, & \text{if } n \in \{0, 1\} \end{cases}.$$

In other words we have simple characterization of the 2-adic valuation of the number $b_n$ for all $n \in \mathbb{N}$.
Let $m \in \mathbb{N}_+$ and consider the series

$$B_m(x) := B(x)^m = \prod_{n=0}^{\infty} \frac{1}{(1 - x^{2n})^m} = \sum_{n=0}^{\infty} b_m(n)x^n.$$
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We have $b_1(n) = b_n$ for $n \in \mathbb{N}$ and

$$b_m(n) = \sum_{i_1 + i_2 + \ldots + i_m = n} \prod_{k=1}^{m} b(i_k),$$

i.e., $b_m(n)$ is Cauchy convolution of $m$-copies of the sequence $(b_n)_{n \in \mathbb{N}}$. For $m \in \mathbb{N}_+$ we denote the sequence $(b_m(n))_{n \in \mathbb{N}}$ by $b_m$. 

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From the above expression we easily deduce that the number \( b_m(n) \) has a natural combinatorial interpretation. Indeed, \( b_m(n) \) counts the number of representations of the integer \( n \) as the sum of powers of 2, where each summand can have one of \( m \) colors.
Now we can formulate the natural

**Question 2**

Let \( m \in \mathbb{N}_+ \) be given. What can be said about the sequence \((\nu_2(b_m(n)))_{n \in \mathbb{N}}\)?
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To give a partial answer to this question we will need two lemmas. The one concerning the characterization of parity of the number $b_m(n)$ and the second one concerning the behaviour of certain binomial coefficients modulo small powers of two.
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**Lemma 1**

Let $m \in \mathbb{N}_+$ be fixed and write $m = 2^k(2u + 1)$ with $k \in \mathbb{N}$. Then:

1. We have $b_m(n) \equiv \binom{m}{n} + 2^{k+1}\binom{m-2}{n-2} \pmod{2^{k+2}}$ for $m$ even;
2. We have $b_m(n) \equiv \binom{m}{n} \pmod{2}$ for $m$ odd;
3. For infinitely many $n$ we have $b_m(n) \not\equiv 0 \pmod{4}$ for $m$ odd.
Lemma 2

Let \( m \) be a positive integer \( \geq 2 \). Then

\[
\binom{2^m - 1}{k} \equiv 1 \pmod{2}, \quad \text{for} \quad k = 0, 1, \ldots, 2^m - 1,
\]

and

\[
\binom{2^m}{k} \equiv \begin{cases} 
1 & \text{for } k = 0, 2^m \\
4 & \text{for } k = 2^{m-2}, 3 \cdot 2^{m-2} \\
6 & \text{for } k = 2^{m-1} \\
0 & \text{in the remaining cases}
\end{cases} \pmod{8}, \quad \text{for} \quad k = 0, 1, \ldots, 2^m.
\]
We are ready to prove the following

**Theorem 3**

Let \( k \in \mathbb{N}_+ \) be given. Then \( \nu_2(b_{2k-1}(n)) = 0 \) for \( n \leq 2^{k-1} \) and

\[
\nu_2(b_{2k-1}(2^k n + i)) = \nu_2(b_1(2n))
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for each \( i \in \{0, \ldots, 2^k - 1\} \) and \( n \in \mathbb{N}_+ \).
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Proof: First of all, let us observe that the second part of Lemma 1 and the first part of Lemma 2 implies that \( b_{2k-1}(n) \) is odd for \( n \leq 2^k - 1 \) and thus \( \nu_2(b_{2k-1}(n)) = 0 \) in this case.
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**Proof:** First of all, let us observe that the second part of Lemma 1 and the first part of Lemma 2 implies that $b_{2^{k-1}}(n)$ is odd for $n \leq 2^k - 1$ and thus $\nu_2(b_{2^{k-1}}(n)) = 0$ in this case.

Let us observe that from the identity $B_{2^{k-1}}(x) = T(x)B_{2^k}(x)$ we get the identity

$$
b_{2^{k-1}}(n) = \sum_{j=0}^{n} t_{n-j} b_{2^k}(j), \quad (1)
$$

where $t_n$ is the $n$-th term of the PTM sequence.
Now let us observe that from the first part of Lemma 1 and the second part of Lemma 2 we have

\[ b_{2^k}(n) \equiv \binom{2^k}{n} \pmod{8} \]

for \( n = 0, 1, \ldots, 2^k \) and \( b_{2^k}(n) \equiv 0 \pmod{8} \) for \( n > 2^k \), provided \( k \geq 2 \) or \( n \neq 2 \).
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Moreover,

\[ b_2(2) \equiv \binom{2}{2} + 4 \binom{0}{0} = 5 \pmod{8}. \]
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Summing up this discussion we have the following expression for \( b_{2^k-1}(n) \pmod{8} \), where \( k \geq 2 \) and \( n \geq 2^k \):

\[
\begin{align*}
 b_{2^k-1}(n) &= \sum_{j=0}^{n} t_{n-j} b_{2^k}(j) = \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) + \sum_{j=2^k+1}^{n} t_{n-j} b_{2^k}(j) \\
 &\equiv \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) \\
 &\equiv \sum_{j=0}^{2^k} t_{n-j} \binom{2^k}{j} \pmod{8} \\
 &\equiv t_n + t_{n-2^k} + 4t_{n-2^k-2} + 4t_{n-3\cdot2^k-2} + 6t_{n-2^k-1} \pmod{8}.
\end{align*}
\]
However, it is clear that $t_{n-2^k-2} + t_{n-3.2^k-2} \equiv 0 \pmod{2}$ and thus we can simplify the above expression and get

$$b_{2^k-1}(n) \equiv t_n + t_{n-2^k} + 6t_{n-2^k-1} \pmod{8}$$

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\]
for \( n \geq 2^k \).

If \( k = 1 \) and \( n \geq 2 \) then, analogously, we get
\[
b_1(n) \equiv \sum_{j=0}^{2^k} t_{n-j}b_{2^k}(j) \pmod{8} \equiv t_n + 5t_{n-2} + 2t_{n-1} \pmod{8}
\]
and since \( t_{n-1} \equiv t_{n-2} \pmod{2} \), we thus conclude that
\[
b_1(n) \equiv t_n + t_{n-2^k} + 6t_{n-2^{k-1}} \pmod{8}.
\]
Let us put
\[ R_k(n) = t_n + t_{n-2^k} + 6t_{n-2^k-1}. \]
Using now the recurrence relations for \( t_n \), i.e., \( t_{2n} = t_n, t_{2n+1} = -t_n \), we easily deduce the identities
\[ R_k(2n) = R_{k-1}(n), \quad R_k(2n + 1) = -R_{k-1}(n) \]
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\[ R_k(n) = t_n + t_{n-2^k} + 6t_{n-2^k-1}. \]
Using now the recurrence relations for \( t_n \), i.e., \( t_{2n} = t_n, t_{2n+1} = -t_n \), we easily deduce the identities
\[ R_k(2n) = R_{k-1}(n), \quad R_k(2n+1) = -R_{k-1}(n) \]
for \( k \geq 2 \).
Using a simple induction argument, one can easily obtain the following identities:
\[ |R_k(2^k m + j)| = |R_1(2m)| \quad (2) \]
for \( k \geq 2, m \in \mathbb{N} \) and \( j \in \{0, \ldots, 2^k - 1\} \).
From the above identity we easily deduce that $R_k(n) \not\equiv 0 \pmod{8}$ for each $n \in \mathbb{N}$ and each $k \geq 1$. If $k = 1$ then $R_1(n) = t_n + 6t_{n-1} + t_{n-2}$ and $R_1(n) \equiv 0 \pmod{8}$ if and only if $t_n = t_{n-1} = t_{n-2}$. However, a well known property of the Prouhet-Thue-Morse sequence is that there are no three consecutive terms which are equal.
From the above identity we easily deduce that $R_k(n) \not\equiv 0 \pmod{8}$ for each $n \in \mathbb{N}$ and each $k \geq 1$. If $k = 1$ then $R_1(n) = t_n + 6t_{n-1} + t_{n-2}$ and $R_1(n) \equiv 0 \pmod{8}$ if and only if $t_n = t_{n-1} = t_{n-2}$. However, a well known property of the Prouhet-Thue-Morse sequence is that there are no three consecutive terms which are equal.

If $k \geq 2$ then our statement about $R_k(n)$ is clearly true for $n \leq 2^k$. If $n > 2^k$ then we can write $n = 2^k m + j$ for some $m \in \mathbb{N}$ and $j \in \{0, 1, \ldots, 2^k - 1\}$. Using the reduction (2) and the property obtained for $k = 1$, we get the result.
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Summing up our discussion, we have proved that $\nu_2(b_{2k-1}(n)) \leq 2$ for each $n \in \mathbb{N}$, since $\nu_2(b_1(n)) \in \{0, 1, 2\}$. Moreover, as an immediate consequence of our reasoning we get the equality

$$\nu_2(b_{2k-1}(2^kn + j)) = \nu_2(b_1(2n))$$

for $j \in \{0, \ldots, 2^k - 1\}$ and our theorem is proved.
Conjecture 1

Let $m \in \mathbb{N}_{\geq 2}$ be given and suppose that $m$ is not of the form $2^k - 1$ for $k \in \mathbb{N}_+$. Then the sequence $(\nu_2(b_m(n)))_{n \in \mathbb{N}}$ is unbounded.
Conjecture 1

Let \( m \in \mathbb{N}_{\geq 2} \) be given and suppose that \( m \) is not of the form \( 2^k - 1 \) for \( k \in \mathbb{N}_+ \). Then the sequence \( (\nu_2(b_m(n)))_{n \in \mathbb{N}} \) is unbounded.

Conjecture 2

Let \( m \) be a fixed positive integer. Then for each \( n \in \mathbb{N} \) and \( k \geq m + 2 \) the following congruence holds

\[
b_{2^m}(2^{k+1}n) \equiv b_{2^m}(2^{k-1}n) \pmod{2^k}.
\]
Conjecture 3

Let $m$ be a fixed positive integer. Then for each $n \in \mathbb{N}$ and $k \geq m + 2$ the following congruence holds

$$b_{2m-1}(2^{k+1}n) \equiv b_{2m-1}(2^k n) \pmod{2^{4\left\lfloor \frac{k+1}{2} \right\rfloor} - 2}.$$
Conjecture 3

Let \( m \) be a fixed positive integer. Then for each \( n \in \mathbb{N} \) and \( k \geq m + 2 \) the following congruence holds

\[
b_{2^m-1}(2^{k+1}n) \equiv b_{2^m-1}(2^{k-1}n) \pmod{2^4 \lfloor \frac{k+1}{2} \rfloor - 2}.
\]

In fact we expect the following

Conjecture 4

Let \( m \) be a fixed positive integer. Then for each \( n \in \mathbb{N} \) and given \( k \gg 1 \) there is a non-decreasing function \( f : \mathbb{N} \to \mathbb{N} \) such that \( f(k) = O(k) \) and the following congruence holds

\[
b_m(2^{k+1}n) \equiv b_m(2^{k-1}n) \pmod{2^f(k)}.
\]
Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of integers and write
\[ f(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n \in \mathbb{Z}[[x]]. \]
Moreover, for \(m \in \mathbb{N}_+\) we define the sequence
\[ b_m = (b_m(n))_{n \in \mathbb{N}}, \]
where
\[ \frac{1}{f(x)^m} = \sum_{n=0}^{\infty} b_m(n) x^n. \]
Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of integers and write
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\[
\frac{1}{f(x)^m} = \sum_{n=0}^{\infty} b_m(n)x^n.
\]

We have the following

**Theorem 4**

Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of integers and suppose that \(\varepsilon_n \equiv 1 \pmod{2}\) for each \(n \in \mathbb{N}\). Then for any \(m \in \mathbb{N}_+\) and \(n \geq m\) we have the congruence
\[
b_{m-1}(n) \equiv \sum_{i=0}^{m} \binom{m}{i} \varepsilon_{n-i} \quad (\text{mod } 2^\nu_2(m)+1). \tag{3}
\]
Proof: Let \( f(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n \in \mathbb{Z}[[x]] \). From the assumption on sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \) we get that

\[
f(x) \equiv \frac{1}{1 + x} \quad (\text{mod } 2).
\]

In consequence, writing \( m = 2^{\nu_2(m)} k \) with \( k \) odd, and using the well known property saying that \( U \equiv V \mod 2^k \) implies \( U^2 \equiv V^2 \mod 2^{k+1} \), we get the congruence

\[
\frac{1}{f(x)^m} \equiv (1 + x)^m \quad (\text{mod } 2^{\nu_2(m)+1}).
\]
Proof: Let $f(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n \in \mathbb{Z}[[x]]$. From the assumption on sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ we get that

$$f(x) \equiv \frac{1}{1+x} \pmod{2}.$$

In consequence, writing $m = 2^{\nu_2(m)} k$ with $k$ odd, and using the well known property saying that $U \equiv V \pmod{2^k}$ implies $U^2 \equiv V^2 \pmod{2^{k+1}}$, we get the congruence

$$\frac{1}{f(x)} \equiv (1 + x)^{m} \pmod{2^{\nu_2(m)+1}}.$$

Thus, multiplying both sides of the above congruence by $f(x)$ we get

$$\frac{1}{f(x)^{m-1}} \equiv f(x)(1 + x)^{m} \pmod{2^{\nu_2(m)+1}}.$$

From the power series expansion of $f(x)(1 + x)^{m}$ by comparing coefficients on the both sides of the above congruence we get that

$$b_{m-1}(n) \equiv \sum_{i=0}^{\min\{m,n\}} \binom{m}{i} \varepsilon_{n-i} \pmod{2^{\nu_2(m)+1}},$$

i.e., for $n \geq m$ we get the congruence (3). Our theorem is proved.
From our result we can deduce the following

**Corollary 5**

Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a non-eventually constant sequence, \(\varepsilon_n \in \{-1, 1\}\) for each \(n \in \mathbb{N}\), and suppose that for each \(N \in \mathbb{N}_+\) there are infinitely many \(n \in \mathbb{N}\) such that \(\varepsilon_n = \varepsilon_{n+1} = \ldots = \varepsilon_{n+N}\). Then, for each even \(m \in \mathbb{N}_+\) there are infinitely many \(n \in \mathbb{N}\) such that

\[
\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1 \quad \text{and} \quad \nu_2(b_{m-1}(n+1)) = 1.
\]
Corollary 5

Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a non-eventually constant sequence, \(\varepsilon_n \in \{-1, 1\}\) for each \(n \in \mathbb{N}\), and suppose that for each \(N \in \mathbb{N}_+\) there are infinitely many \(n \in \mathbb{N}\) such that \(\varepsilon_n = \varepsilon_{n+1} = \ldots = \varepsilon_{n+N}\). Then, for each even \(m \in \mathbb{N}_+\) there are infinitely many \(n \in \mathbb{N}\) such that

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\]

Proof: From our assumption on the sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) we can find infinitely many \((m + 1)\)-tuples such that \(\varepsilon_{n+1} = \varepsilon, \varepsilon_n = \ldots = \varepsilon_{n-m} = -\varepsilon\), where \(\varepsilon\) is a fixed element of \(\{-1, 1\}\). We apply (3) and get

\[
b_{m-1}(n) \equiv \sum_{i=0}^{m} \binom{m}{i} \varepsilon_{n-i} \equiv -\sum_{i=0}^{m} \binom{m}{i} \varepsilon \equiv -\varepsilon 2^m \equiv 0 \pmod{2^{\nu_2(m)+1}},
\]

\[
b_{m-1}(n+1) \equiv \sum_{i=0}^{m} \binom{m}{i} \varepsilon_{n+1-i} \equiv 2\varepsilon - \sum_{i=0}^{m} \binom{m}{i} \varepsilon \equiv \varepsilon(2 - 2^m) \equiv 2\varepsilon \pmod{2^{\nu_2(m)+1}}.
\]
From our result we can deduce the following

**Corollary 5**

Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a non-eventually constant sequence, \(\varepsilon_n \in \{-1, 1\}\) for each \(n \in \mathbb{N}\), and suppose that for each \(N \in \mathbb{N}_+\) there are infinitely many \(n \in \mathbb{N}\) such that \(\varepsilon_n = \varepsilon_{n+1} = \ldots = \varepsilon_{n+N}\). Then, for each even \(m \in \mathbb{N}_+\) there are infinitely many \(n \in \mathbb{N}\) such that

\[
\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1 \quad \text{and} \quad \nu_2(b_{m-1}(n+1)) = 1.
\]

**Proof:** From our assumption on the sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) we can find infinitely many \((m+1)\)-tuples such that \(\varepsilon_{n+1} = \varepsilon, \varepsilon_n = \ldots = \varepsilon_{n-m} = -\varepsilon\), where \(\varepsilon\) is a fixed element of \([-1, 1]\). We apply (3) and get

\[
b_{m-1}(n) \equiv \sum_{i=0}^{m} \binom{m}{i} \varepsilon_{n-i} \equiv -\sum_{i=0}^{m} \binom{m}{i} \varepsilon \equiv -\varepsilon 2^m \equiv 0 \pmod{2^{\nu_2(m)+1}},
\]

\[
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\]

In consequence \(\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1\) and \(\nu_2(b_{m-1}(n+1)) = 1\) and our theorem is proved.
Example: Let $F : \mathbb{N} \to \mathbb{N}$ satisfy the condition 
$$\limsup_{n \to +\infty} (F(n+1) - F(n)) = +\infty$$
and define the sequence 
$$\varepsilon_n(F) = \begin{cases} 
1 & n = F(m) \text{ for some } m \in \mathbb{N} \\
-1 & \text{otherwise} 
\end{cases}.$$
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\[
\varepsilon_n(F) = \begin{cases} 
1 & n = F(m) \text{ for some } m \in \mathbb{N} \\
-1 & \text{otherwise}
\end{cases}.
\]

It is clear that the sequence $(\varepsilon_n(F))_{n \in \mathbb{N}}$ satisfies the conditions from Theorem 5 and thus for any even $m \in \mathbb{N}_+$ there are infinitely many $n \geq m$ such that $\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1$ and $\nu_2(b_{m-1}(n + 1)) = 1$. 

A particular examples of $F$'s satisfying required properties include:
- positive polynomials of degree $\geq 2$;
- the functions which for given $n \in \mathbb{N}_+$ take as value the $n$-th prime number of the form $ak + b$, where $a \in \mathbb{N}_+$, $b \in \mathbb{Z}$ and $\gcd(a, b) = 1$;
- and many others.
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\]

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- and many others.
Lemma 6

Let $s \in \mathbb{N}_{\geq 3}$. Then

\[
\binom{2^s}{i} \pmod{16} \equiv \begin{cases}  
1 & \text{for } i = 0, 2^s \\
6 & \text{for } i = 2^{s-1} \\
8 & \text{for } i = (2j + 1)2^{s-3}, j \in \{0, 1, 2, 3\} \\
12 & \text{for } i = 2^{s-2}, 3 \cdot 2^{s-2} \\
0 & \text{in the remaining cases}
\end{cases}.
\]
Theorem 7

Let \( s \in \mathbb{N}_+ \) and \((\varepsilon_n)_{n \in \mathbb{N}}\) be an integer sequence and suppose that \( \varepsilon_n \equiv 1 \pmod{2} \) for \( n \in \mathbb{N} \).

(A) For \( n \geq 2^s \) we have

\[
b_{2^s-1}(n) \equiv \varepsilon_n + 2\varepsilon_{n-2^s-1} + \varepsilon_{n-2^s} \pmod{4}.
\]

In particular, if \( \varepsilon_n \in \{-1, 1\} \) for all \( n \in \mathbb{N} \) then:

\[
\nu_2(b_{2^s-1}(n)) > 1 \iff \varepsilon_n = \varepsilon_{n-2^s-1} = \varepsilon_{n-2^s} \text{ or } \varepsilon_n = -\varepsilon_{n-2^s-1} = \varepsilon_{n-2^s}
\]

\[
\nu_2(b_{2^s-1}(n)) = 1 \iff \varepsilon_n = -\varepsilon_{n-2^s}.
\]

(B) For \( s \geq 2 \) and \( n \geq 2^s \) we have

\[
b_{2^s-1}(n) \equiv \varepsilon_n + 6\varepsilon_{n-2^s-1} + \varepsilon_{n-2^s} \pmod{8}.
\]

In particular, if \( \varepsilon_n \in \{-1, 1\} \) for all \( n \in \mathbb{N} \), then:

\[
\nu_2(b_{2^s-1}(n)) > 2 \iff \varepsilon_n = \varepsilon_{n-2^s-1} = \varepsilon_{n-2^s}
\]

\[
\nu_2(b_{2^s-1}(n)) = 2 \iff \varepsilon_n = -\varepsilon_{n-2^s-1} = \varepsilon_{n-2^s}
\]

\[
\nu_2(b_{2^s-1}(n)) = 1 \iff \varepsilon_n = -\varepsilon_{n-2^s}.
\]
Theorem 7 (continuation)

For \( s \geq 3 \) and \( n \geq 2^s \) we have

\[
b_{2^s-1}(n) \equiv \varepsilon_n + \varepsilon_{n-2^s} + 6\varepsilon_{n-2^s-1} + 12(\varepsilon_{n-2^s-2} + \varepsilon_{n-3\cdot2^s-2}) \pmod{16} \tag{6}
\]

In particular, if \( \varepsilon_n \in \{-1, 1\} \) for all \( n \in \mathbb{N} \), then:

\[
\nu_2(b_{2^s-1}(n)) > 3 \iff \varepsilon_n = \varepsilon_{n-2^s-2} = \varepsilon_{n-2^s-1} = \varepsilon_{n-3\cdot2^s-2} = \varepsilon_{n-2^s} \text{ or }
\varepsilon_n = -\varepsilon_{n-2^s-2} = \varepsilon_{n-2^s-1} = -\varepsilon_{n-3\cdot2^s-2} = \varepsilon_{n-2^s};
\]

\[
\nu_2(b_{2^s-1}(n)) = 3 \iff \varepsilon_n = \varepsilon_{n-2^s-2} = \varepsilon_{n-2^s-1} = -\varepsilon_{n-3\cdot2^s-2} = \varepsilon_{n-2^s} \text{ or }
\varepsilon_n = -\varepsilon_{n-2^s-2} = \varepsilon_{n-2^s-1} = \varepsilon_{n-3\cdot2^s-2} = \varepsilon_{n-2^s}
\]

\[
\iff \varepsilon_n \equiv -\varepsilon_{n-2^s} + 2\varepsilon_{n-2^s-1} + 8 \pmod{16} \tag{7}
\]
Theorem 7 (continuation)

(C) For \( s \geq 3 \) and \( n \geq 2^s \) we have

\[
b_{2^s-1}(n) \equiv \varepsilon_n + \varepsilon_{n-2^s} + 6 \varepsilon_{n-2^s-1} + 12(\varepsilon_{n-2^s-2} + \varepsilon_{n-3\cdot2^s-2}) \pmod{16} \tag{6}
\]

In particular, if \( \varepsilon_n \in \{-1, 1\} \) for all \( n \in \mathbb{N} \), then:

\[
\nu_2(b_{2^s-1}(n)) > 3 \iff \varepsilon_n = \varepsilon_{n-2^s-2} = \varepsilon_{n-2^s-1} = \varepsilon_{n-3\cdot2^s-2} = \varepsilon_{n-2^s} \quad \text{or} \quad \varepsilon_n = -\varepsilon_{n-2^s-2} = \varepsilon_{n-2^s-1} = -\varepsilon_{n-3\cdot2^s-2} = \varepsilon_{n-2^s};
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\]

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\]

As a first application of Theorem 17 we get the following:

Corollary 8

Let \( s \in \mathbb{N}_{\geq 2} \) and \( (\varepsilon_n)_{n \in \mathbb{N}} \) with \( \varepsilon_n \in \{-1, 1\} \) for all \( n \in \mathbb{N} \). If there is no \( n \in \mathbb{N}_{\geq 2^s} \) such that \( \varepsilon_n = \varepsilon_{n-2^s-1} = \varepsilon_{n-2^s} \) then

\[
\nu_2(b_{2^s-1}(n)) = \nu_2(\varepsilon_n + 6\varepsilon_{n-2^s-1} + \varepsilon_{n-2^s}).
\]

In particular, for each \( n \in \mathbb{N}_{\geq 2^s} \) we have \( \nu_2(b_{2^s-1}(n)) \in \{1, 2\} \).
We consider now the power series

\[ F_1(x) = \frac{1}{1 - x} \prod_{n=0}^{\infty} \frac{1}{1 - x^{2^n}} = \sum_{n=0}^{\infty} b_{2n} x^n, \]

where \( b_n \) is the binary partition function.
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where \( b_n \) is the binary partition function.

Let \( m \in \mathbb{Z} \) and write

\[ F_m(x) = F_1(x)^m = \frac{1}{(1 - x)^m} \prod_{n=0}^{\infty} \frac{1}{(1 - x^{2^n})^m} = \sum_{n=0}^{\infty} c_m(n) x^n. \]

If \( m \in \mathbb{N}_+ \), then the sequence \( (c_m(n))_{n \in \mathbb{N}} \), has a natural combinatorial interpretation. More precisely, the number \( c_m(n) \) counts the number of binary representations of \( n \) such that the part equal to 1 can take one among \( 2^m \) colors and other parts can have \( m \) colors. Motivated by the mentioned result concerning the 2-adic valuation of the number \( b_m(n) \), it is natural to ask about the behaviour of the sequence \( (\nu_2(c_m(n)))_{n \in \mathbb{N}}, m \in \mathbb{Z} \).
Let us observe the identity $F_1(x) = \frac{1}{1-x} B(x)$. Thus, the functional relation $(1 - x)B(x) = B(x^2)$ implies the functional relation $(1 - x)F_1(x) = (1 + x)F_1(x^2)$ for the series $F_1$. In consequence, for $m \in \mathbb{Z}$ we have the relation

$$F_m(x) = \left(\frac{1 + x}{1 - x}\right)^m F_m(x^2),$$

which will be useful later.
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$$F_m(x) = \left(\frac{1 + x}{1 - x}\right)^m F_m(x^2),$$

which will be useful later.

In the sequel we will need the following functional property: for $m_1, m_2 \in \mathbb{Z}$ we have

$$F_{m_1}(x)F_{m_2}(x) = F_{m_1+m_2}(x).$$
We start our investigations with the simple lemma which is a consequence of the result of Churchhouse and the product form of the series $F_{-1}(x)$.

**Lemma 9**

For $n \in \mathbb{N}_+$, we have the following equalities:

\[
\nu_2(c_1(n)) = \frac{1}{2} |t_n + 3t_{n-1}|,
\]

\[
\nu_2(c_{-1}(n)) = \begin{cases} 
1, & \text{if } t_n \neq t_{n-1} \\
+\infty, & \text{if } t_n = t_{n-1} 
\end{cases}
\]

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$p$-adic valuations ...
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1, & \text{if } t_n \neq t_{n-1} \\
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\end{cases}.
\]

Proof: The first equality is an immediate consequence of the equalities $c_1(n) = b(2n), \nu_2(b(n)) = \frac{1}{2}|t_n - 2t_{n-1} + t_{n-2}|$ and the recurrence relations satisfied by the PTM sequence $(t_n)_{n \in \mathbb{N}}$, i.e., $t_{2n} = t_n, t_{2n+1} = -t_n$. The second equality comes from the expansion

\[
F_{-1}(x) = (1 - x) \prod_{n=0}^{\infty} (1 - x^{2^n}) = (1 - x) \sum_{n=0}^{\infty} t_n x^n = 1 + \sum_{n=1}^{\infty} (t_n - t_{n-1})x^n.
\]
In order to compute the 2-adic valuations of the sequence \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) we need the following simple

**Lemma 10**

The sequence \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) satisfy the following recurrence relations:
\[c_{\pm 2}(0) = 1, \quad c_{\pm 2}(1) = \pm 4 \quad \text{and for } n \geq 1 \quad \text{we have}\]
\[c_{\pm 2}(2n) = \pm 2c_{\pm 2}(2n - 1) - c_{\pm 2}(2n - 2) + c_{\pm 2}(n) + c_{\pm 2}(n - 1),\]
\[c_{\pm 2}(2n + 1) = \pm 2c_{\pm 2}(2n) - c_{\pm 2}(2n - 1) \pm 2c_{\pm 2}(n).\]
In order to compute the 2-adic valuations of the sequence \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) we need the following simple

**Lemma 10**

*The sequence \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) satisfy the following recurrence relations:*
\[c_{\pm 2}(0) = 1, \quad c_{\pm 2}(1) = \pm 4 \text{ and for } n \geq 1 \text{ we have}\]
\[
c_{\pm 2}(2n) = \pm 2c_{\pm 2}(2n - 1) - c_{\pm 2}(2n - 2) + c_{\pm 2}(n) + c_{\pm 2}(n - 1),
\]
\[
c_{\pm 2}(2n + 1) = \pm 2c_{\pm 2}(2n) - c_{\pm 2}(2n - 1) \pm 2c_{\pm 2}(n).
\]

Proof: The recurrence relations for the sequence \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) are immediate consequence of the functional equation
\[
F_{\pm 2}(x) = \left(\frac{1+2x}{1-x}\right)^{\pm 2} F_{\pm 2}(x^2),
\]
which can be rewritten in an equivalent form
\[
(1 - x)^{\pm 2} F_{\pm 2}(x) = (1 + x)^{\pm 2} F_{\pm 2}(x^2).
\]
Comparing now the coefficients on both sides of this relation we get the result.
As a consequence of the recurrence relations for \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) we get

**Corollary 11**

*For \(n \in \mathbb{N}_+\) we have \(c_{\pm 2}(n) \equiv 4 \pmod{8}\). In consequence, for \(n \in \mathbb{N}_+\) we have \(\nu_2(c_{\pm 2}(n)) = 2\).*
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Proof: The proof relies on a simple induction. Indeed, we have \(c_{\pm 2}(1) = \pm 4\), \(c_{-2}(2) = 4\), \(c_2(2) = 12\) and thus our statement folds for \(n = 1, 2\). Assuming it holds for all integers \(\leq n\) and applying the recurrence relations given in Lemma 10 we get the result.

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As a consequence of the recurrence relations for \((c_{\pm 2}(n))_{n \in \mathbb{N}}\) we get

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**Proof:** The proof relies on a simple induction. Indeed, we have \(c_{\pm 2}(1) = \pm 4, c_{-2}(2) = 4, c_2(2) = 12\) and thus our statement folds for \(n = 1, 2\). Assuming it holds for all integers \(\leq n\) and applying the recurrence relations given in Lemma 10 we get the result.

The second part is an immediate consequence of the obtained congruence.

**Theorem 12**

*Let* \(m \in \mathbb{Z} \setminus \{0, -1\}\) *and consider the sequence* \(c_m = (c_m(n))_{n \in \mathbb{N}}\). *Then* \(c_m(0) = 1\) *and for* \(n \in \mathbb{N}_+\) *we have*

\[
\nu_2(c_m(n)) = \begin{cases} 
\nu_2(m) + 1, & \text{if } m \equiv 0 \pmod{2} \\
1, & \text{if } m \equiv 1 \pmod{2} \text{ and } t_n \neq t_{n-1} \\
\nu_2(m + 1) + 1, & \text{if } m \equiv 1 \pmod{2} \text{ and } t_n = t_{n-1}
\end{cases}
\]

(8)
Proof: First of all, let us note that our theorem is true for $m = 1, \pm 2$. This is a consequence of Lemma 9 and Corollary 11. Let $m \in \mathbb{Z}$ and $|m| > 2$. Because $c_m(0) = 1, c_m(1) = 2m$ our statement is clearly true for $n = 0, 1$. We can assume that $n \geq 2$. 
Proof: First of all, let us note that our theorem is true for $m = 1, \pm 2$. This is a consequence of Lemma 9 and Corollary 11. Let $m \in \mathbb{Z}$ and $|m| > 2$. Because $c_m(0) = 1$, $c_m(1) = 2m$ our statement is clearly true for $n = 0, 1$. We can assume that $n \geq 2$.

We start with the case $m = -3$. From the functional relation $F_{-3}(x) = F_{-2}(x)F_{-1}(x)$ we immediately get the identity

$$c_{-3}(n) = \sum_{i=0}^{n} c_{-1}(i)c_{-2}(n-i) = c_{-2}(n)+t_n-t_{n-1}+\sum_{i=1}^{n-1}(t_i-t_{i-1})c_{-2}(n-i).$$

Let us observe that for $i \in \{1, \ldots, n-1\}$, from Lemma 9 and Corollary 11, we obtain the inequality

$$\nu_2((t_i-t_{i-1})c_{-2}(n-i)) \geq 3.$$
Proof: First of all, let us note that our theorem is true for $m = 1, \pm 2$. This is a consequence of Lemma 9 and Corollary 11. Let $m \in \mathbb{Z}$ and $|m| > 2$. Because $c_m(0) = 1, c_m(1) = 2m$ our statement is clearly true for $n = 0, 1$. We can assume that $n \geq 2$.

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$$c_{-3}(n) = \sum_{n=0}^{n} c_{-1}(i)c_{-2}(n-i) = c_{-2}(n) + t_n - t_{n-1} + \sum_{i=1}^{n-1} (t_i - t_{i-1})c_{-2}(n-i).$$

Let us observe that for $i \in \{1, \ldots, n-1\}$, from Lemma 9 and Corollary 11, we obtain the inequality

$$\nu_2((t_i - t_{i-1})c_{-2}(n-i)) \geq 3.$$

In consequence, from Lemma 10, we get

$$c_{-3}(n) \equiv c_{-2}(n) + t_n - t_{n-1} \equiv 4 + t_n - t_{n-1} \pmod{8}.$$

It is clear that $4 + t_n - t_{n-1} \not\equiv 0 \pmod{8}$. Thus, we get the equality $\nu_2(c_{-3}(n)) = \nu_2(4 + t_n - t_{n-1})$ and the result follows for $m = -3$. 

Maciej Ulas
We are ready to prove the general result. We proceed by double induction on $m$ (which depends on the remainder of $m$ (mod 4)) and $n \in \mathbb{N}_+$. As we already proved, our theorem is true for $m = \pm 1, \pm 2$ and $m = -3$. Let us assume that it is true for each $m$ satisfying $|m| < M$ and each term $c_m(j)$ with $j < n$. Let $|m| \geq M$ and write $m = 4k + r$ with $|k| < M/4$ for some $r \in \{-3, -2, 0, 1, 2, 3\}$ (depending on the sign of $m$).
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If $m = 4k$, then from the identity $F_{4k}(x) = F_{2k}(x)^2$ we get the expression

$$c_{4k}(n) = 2c_{2k}(n) + \sum_{i=1}^{n-1} c_{2k}(i)c_{2k}(n - i).$$

From the induction hypothesis we have

$$\nu_2(c_{2k}(i)c_{2k}(n - i)) = 2(\nu_2(2k) + 1) > \nu_2(2c_{2k}(n)) = \nu_2(2k) + 2.$$ In consequence

$$\nu_2(c_m(n)) = \nu_2(c_{4k}(n)) = \nu_2(2c_{2k}(n)) = \nu_2(2k) + 2 = \nu_2(4k) + 1.$$ The obtained equality finishes the proof in the case $m \equiv 0 \pmod{4}$. 

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\]

In consequence
\[
\nu_2(c_m(n)) = \nu_2(c_{4k}(n)) = \nu_2(2c_{2k}(n)) = \nu_2(2k) + 2 = \nu_2(4k) + 1.
\]

The obtained equality finishes the proof in the case \( m \equiv 0 \) (mod 4).

Similarly, if \( m = 4k + 2 \) is positive, we use the identity \( F_{4k+2}(x) = F_{4k}(x)F_2(x) \), and get

\[
c_{4k+2}(n) = c_2(n) + c_{4k}(n) + \sum_{i=1}^{n-1} c_{4k}(i)c_2(n-i).
\]
From the equalities \( \nu_2(c_2(n)) = \nu_2(2) + 1 \) and
\( \nu_2(c_{4k}(n)) = \nu_2(4k) + 1, n \in \mathbb{N}_+ \),
we get \( \nu_2(c_{4k}(i)c_2(n - i)) = \nu_2(k) + 5 \)
for each \( i \in \{1, \ldots, n-1\} \).
Thus
\( \nu_2(c_2(n) + c_{4k}(n)) = \nu_2(c_2(n)) = 2 = \nu_2(4k + 2) + 1. \)
From the equalities $\nu_2(c_2(n)) = \nu_2(2) + 1$ and $\nu_2(c_{4k}(n)) = \nu_2(4k) + 1$, $n \in \mathbb{N}_+$, we get $\nu_2(c_{4k}(i)c_2(n - i)) = \nu_2(k) + 5$ for each $i \in \{1, \ldots, n - 1\}$. Thus $\nu_2(c_2(n) + c_{4k}(n)) = \nu_2(c_2(n)) = 2 = \nu_2(4k + 2) + 1$.

If $m = 4k + 2$ is negative, we use the identity $F_{4k+2}(x) = F_{4(k+1)}(x)F_{-2}(x)$ and proceed in exactly the same way.
From the equalities \( \nu_2(c_2(n)) = \nu_2(2) + 1 \) and 
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\( \nu_2(c_{4k}(i)c_2(n - i)) = \nu_2(k) + 5 \)
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\( F_{4k+2}(x) = F_{4(k+1)}(x)F_{-2}(x) \)
and proceed in exactly the same way.

If \( m = 4k + 1 > 0 \), then we use the identity 
\( F_{4k+1}(x) = F_{4k}(x)F_1(x) \) and get
\[
c_{4k+1}(n) = c_{4k}(n) + c_1(n) + \sum_{i=1}^{n-1} c_{4k}(i)c_1(n - i).
\]

From induction hypothesis we have 
\( \nu_2(c_{4k}(i)c_1(n - i)) \geq \nu_2(4k) + 2 \geq 4. \)
Moreover, for \( n \in \mathbb{N}_+ \) we have 
\( \nu_2(c_1(n)) \in \{1, 2\} \). Thus
\[
\nu_2(c_{4k}(n) + c_1(n)) = \nu_2(c_1(n)) = \begin{cases} 
1, & \text{if } t_n \neq t_{n-1} \\
2, & \text{if } t_n = t_{n-1}
\end{cases}.
\]
as we claimed.
If $m = 4k + 1 < 0$, we write $m = 4(k + 1) - 3$ and use the identity $F_{4k+1}(x) = F_{4(k+1)}(x)F_{-3}(x)$. Next, using the obtained expression for $\nu_2(c_{-3}(n))$ and $\nu_2(c_{4(k+1)}(n))$ and the same reasoning as in the positive case we get the result.
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Finally, if $m = 4k + 3 > 0$ we use the identity $F_{4k+3}(x) = F_{4(k+1)}(x)F_{-1}(x)$ which leads us to the expression

$$c_{4k+3}(n) = c_{4k}(n) + c_{-1}(n) + \sum_{i=1}^{n-1} c_{4k}(i)c_{-1}(n - i).$$

It is clear that $\nu_2(c_{4k}(i)c_{-1}(n - i)) > \nu_2(c_{4k}(n) + c_{-1}(n))$ for each $n \in \mathbb{N}_+$ and $i \in \{1, \ldots, n - 1\}$. In consequence, by induction hypothesis

$$\nu_2(c_{4k+3}(n)) = \nu_2(c_{4(k+1)}(n) + c_{-1}(n))$$

$$= \begin{cases} 
1, & \text{if } t_n \neq t_{n-1} \\
\nu_2(c_{4(k+1)}(n)), & \text{if } t_n = t_{n-1} 
\end{cases}$$

$$= \begin{cases} 
1, & \text{if } t_n \neq t_{n-1} \\
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If \( m = 4k + 1 < 0 \), we write \( m = 4(k + 1) - 3 \) and use the identity 
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\[
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\]

It is clear that \( \nu_2(c_{4k}(i)c_{-1}(n - i)) > \nu_2(c_{4k}(n) + c_{-1}(n)) \) for each \( n \in \mathbb{N}^+ \) and \( i \in \{1, \ldots, n - 1\} \). In consequence, by induction hypothesis

\[
\nu_2(c_{4k+3}(n)) = \nu_2(c_{4(k+1)}(n) + c_{-1}(n))
\]

\[
= \begin{cases} 
1, & \text{if } t_n \neq t_{n-1} \\
\nu_2(c_{4(k+1)}(n)), & \text{if } t_n = t_{n-1}
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\]

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\]

If \( m = 4k + 3 < 0 \), then we write \( 4k + 3 = 4(k + 1) - 1 \) and employ the identity 
\[ F_{4k+3}(x) = F_{4(k+1)}(x)F_{-1}(x). \]
Let $n \in \mathbb{N}_+$ and write

$$n = \sum_{i=0}^{k} \varepsilon_i 2^i,$$

where $\varepsilon_i \in \{0, 1\}$ and $k \leq \log_2 n$. The above representation is just the (unique) binary expansion of $n$ in base 2. Let us observe that the equality $\nu_2(n) = u$ implies $\varepsilon_0 = \ldots = \varepsilon_{u-1} = 0$ and $\varepsilon_u = 1$ in the above representation. Thus, if $m \in \mathbb{Z} \setminus \{-1\}$ is fixed, our result concerning the exact value of $\nu_2(c_m(n))$ given by Theorem 16 implies that the number of trailing zeros in the binary expansion of $c_m(n)$, $n \in \mathbb{N}_+$, is bounded.
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This observation suggests the question whether the index of the next non-zero digit in the binary expansion in \( c_m(n) \) is in bounded distance from the first one. We state this in equivalent form as the following

**Question 3**

*Does there exists \( m \in \mathbb{Z} \setminus \{-1\} \) such that the sequence

\[
\left( \nu_2 \left( \frac{c_m(n)}{2^{\nu_2(c_m(n))}} - 1 \right) \right)_{n \in \mathbb{N}}
\]

has finite set of values?*
Let us write $d_m(n) = \nu_2 \left( \frac{c_m(n)}{2^{\nu_2(c_m(n))}} - 1 \right)$. We performed numerical computations for $m \in \mathbb{Z}$ satisfying $|m| < 100$ and $n \leq 10^5$. In this range there are many values of $m$ such that the cardinality of the set of values of the sequence $(d_m(n))_{n \in \mathbb{N}}$ is \leq 4. We define:

$$M_m(x) := \max\{d_m(n) : n \leq x\}, \quad L_m(x) := |\{d_m(n) : n \leq x\}|.$$

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<th>$L_m(10^5)$</th>
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Our numerical computations strongly suggest that there should be infinitely many \( m \in \mathbb{Z} \) such that the sequence \((d_m(n))_{n \in \mathbb{N}}\) is bounded. We even dare to formulate the following

**Conjecture 5**

Let \( k \in \mathbb{N}_+ \) and \( m = 2^{2k} - 1 \). Then the sequence \((d_m(n))_{n \in \mathbb{N}}\) is bounded.

In fact, we expect that for \( n \in \mathbb{N} \) the inequality \( d_{2^{2k} - 1}(n) \leq 2k \) is true.
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It is well known that if $k \in \mathbb{N}_+$ and $t \equiv 1 \pmod{2}$, then

\[
c_1(2^{2k+1}t) - c_1(2^{2k-1}t) \equiv 0 \pmod{2^{3k+2}},
\]

\[
c_1(2^{2k}t) - c_1(2^{2k-2}t) \equiv 0 \pmod{2^{3k}}
\]

(remember $c_1(n) = b(2n)$, where $b(n)$ counts the binary partitions of $n$). The above congruences were conjectured by Churchhouse and independently proved by Rødseth and Gupta. Moreover, there is no higher power of 2 which divides $c_1(4n) - c_1(n)$. 

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*p-adic valuations...*
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It is well known that if \( k \in \mathbb{N}_+ \) and \( t \equiv 1 \pmod{2} \), then

\[
\begin{align*}
c_1(2^{2k+1}t) - c_1(2^{2k-1}t) & \equiv 0 \pmod{2^{3k+2}}, \\
c_1(2^{2k}t) - c_1(2^{2k-2}t) & \equiv 0 \pmod{2^{3k}}
\end{align*}
\]

(remember \( c_1(n) = b(2n) \), where \( b(n) \) counts the binary partitions of \( n \)).

The above congruences were conjectured by Churchhouse and independently proved by Rødseth and Gupta. Moreover, there is no higher power of 2 which divides \( c_1(4n) - c_1(n) \).

This result motivates the question concerning the divisibility of the number \( c_m(2^{k+2}n) - c_m(2^k n) \) by powers of 2. We performed some numerical computations in case of \( m \in \{2, 3, \ldots, 10\} \) and \( n \leq 10^5 \) and believe that the following is true.
Conjecture 6

For $k \in \mathbb{N}_+$ and each $n \in \mathbb{N}_+$, we have:

$$\nu_2(c_{2k}(4n) - c_{2k}(n)) = \nu_2(n) + 2\nu_2(k) + 3.$$ 

Moreover, for $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$ the following inequalities holds

$$\nu_2(c_{4k+1}(4n) - c_{4k+1}(n)) \geq \nu_2(n) + 3,$$

$$\nu_2(c_{4k+3}(4n) - c_{4k+3}(n)) \geq \nu_2(n) + 6.$$ 

In each case the equality holds for infinitely many $n \in \mathbb{N}$. 
Some results for $p$-ary colored partitions

For $k \in \mathbb{N}^+$ we define the sequence $(A_{m,k}(n))_{n \in \mathbb{N}}$, where

$$F_m(x)_k = \prod_{n=0}^{\infty} (1 - x^{mk})^k = \sum_{n=0}^{\infty} A_{m,k}(n)x^n.$$

The sequence $(A_{m,k}(n))_{n \in \mathbb{N}}$, as the sequences considered earlier, can be interpreted in a natural combinatorial way. More precisely, the number $A_{m,k}(n)$ counts the number of representations of $n$ as sums of powers of $m$, where each summand has one among $k$ colors.

A question arises: is it possible to find a simple expression for an exponent $k$, such that the sequence $(\nu_p(A_{p,k}(n)))_{n \in \mathbb{N}}$ is bounded or even can be described in simple terms? Here $p$ is a fixed prime number.
For $k \in \mathbb{N}_+$ we define the sequence $(A_{m,k}(n))_{n \in \mathbb{N}}$, where

$$F_m(x)^k = \prod_{n=0}^{\infty} \frac{1}{(1 - x^{m^n})^k} = \sum_{n=0}^{\infty} A_{m,k}(n)x^n.$$
Some results for \( p \)-ary colored partitions

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Some results for $p$-ary colored partitions

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A question arises: is it possible to find a simple expression for an exponent $k$, such that the sequence $(\nu_p(A_{p,k}(n)))_{n \in \mathbb{N}}$ is bounded or even can be described in simple terms? Here $p$ is a fixed prime number.
For a given $p$ (non-necessarily a prime), an integer $n$ and $i \in \{0, \ldots, p - 1\}$ we define

$$N_p(i, n) = |\{j : n = \sum_{j=0}^{k} \varepsilon_j p^j, \varepsilon_j \in \{0, \ldots, p - 1\} \text{ and } \varepsilon_j = i\}|.$$
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The above number counts the number of the digits equal to $i$ in the base $p$ representation of the integer $n$. From the definition, we immediately deduce the following equalities:

$$N_p(i, 0) = 0, \quad N_p(i, pn + j) = \begin{cases} N_p(i, n), & \text{if } j \neq i \\ N_p(i, n) + 1, & \text{if } j = i \end{cases} \quad (9)$$
We have the following result

**Lemma 13**

Let \( r \in \{1, \ldots, p - 1\} \). We have

\[
F_p(x)^{-r} = \prod_{n=0}^{\infty} (1 - x^{p^n})^r = \sum_{n=0}^{\infty} D_{p,r}(n)x^n,
\]

where

\[
D_{p,r}(n) = \prod_{i=0}^{p-1} (-1)^{iN_p(i,n)} \binom{r}{i}^{N_p(i,n)},
\]

with the convention that \( \binom{a}{b} = 0 \) for \( b > a \) and \( 0^0 = 1 \). Moreover, for \( j \in \{0, \ldots, p - 1\} \) and \( n \in \mathbb{N}_+ \) we have

\[
D_{p,r}(pn + j) = (-1)^j \binom{r}{j} D_{p,r}(n).
\]
Our next result is the following

**Lemma 14**

Let $k \in \mathbb{N}_+$ and suppose that $p - 1 \mid k$. Then

$$F_p(x)^k \equiv (1 - x)^{\frac{k}{p-1}} \pmod{p^{\nu_p(k)+1}}.$$
Our next result is the following

**Lemma 14**

Let $k \in \mathbb{N}_+$ and suppose that $p - 1 | k$. Then

$$F_p(x)^k \equiv (1 - x)^{\frac{k}{p-1}} \pmod{p^{\nu_p(k)+1}}.$$ 

We are ready to present the crucial lemma which is the main tool in our study of the $p$-adic valuation of the number $A_{p,(p-1)(up^{s}-1)}(n)$ in the sequel. More precisely, the lemma contains information about behaviour of the $p$-adic valuation of the expression

$$\sum_{i=0}^{u} (-1)^i \binom{u}{i} D_p(n - i),$$

where

$$D_p(n) := D_{p,p-1}(n).$$

In particular $D_p(n) \neq 0$ for all $n \in \mathbb{N}$. 
Lemma 15

Let $p \geq 3$ be prime and $u \in \{1, \ldots, p - 1\}$. Let $n \geq p$ be of the form

$$n = n'' p^{s+1} + kp^s + j$$

for some $n'' \in \mathbb{N}$, $k \in \{1, \ldots, p - 1\}$, $s \in \mathbb{N}_+$ and $j \in \{0, \ldots, p - 1\}$. Then the following equality holds:

$$\nu_p \left( \sum_{i=0}^{u} (-1)^i \binom{u}{i} D_p(n - i) \right) = \nu_p \left( (p - k) \binom{p + u - 1}{j} + k \binom{p + u - 1}{p + j} \right).$$

In particular:

(a) If $u = 1$, then

$$\nu_p \left( \sum_{i=0}^{u} (-1)^i \binom{u}{i} D_p(n - i) \right) = \nu_p(D_p(n) - D_p(n - 1)) = 1,$$

for any $n \in \mathbb{N}_+$.

(b) If $j \geq u$, then we have the equality

$$\nu_p \left( \sum_{i=0}^{u} (-1)^i \binom{u}{i} D_p(n - i) \right) = 1.$$

(c) If $u \geq 2$, then there exist $j, k \in \{0, \ldots, p - 1\}, k \neq 0$, such that we have

$$\nu_p \left( \sum_{i=0}^{u} (-1)^i \binom{u}{i} D_p(n - i) \right) \geq 2.$$
Theorem 16

Let \( p \in \mathbb{P}_{\geq 3}, u \in \{1, \ldots, p - 1\} \) and \( s \in \mathbb{N}_+ \).

(a) If \( n > up^s \), then
\[
\nu_p(A_{p,(p-1)(up^s-1)}(n)) \geq 1.
\]

(b) If \( n > p^s \), then
\[
\nu_p(A_{p,(p-1)(p^s-1)}(n)) = 1.
\]

(c) If \( u \geq 2 \), then
\[
\nu_p(A_{p,(p-1)(up^s-1)}(n)) = 1
\]
for infinitely many \( n \).

(d) If \( u \geq 2 \), then
\[
\nu_p(A_{p,(p-1)(up^s-1)}(n)) \geq 2
\]
for infinitely many \( n \).

(e) If \( s \geq 2 \) and \( n \geq p^{s+1} \) with the unique base \( p \)-representation \( n = \sum_{i=0}^\nu \varepsilon_i p^i \) and
\[
\nu_p(A_{p,(p-1)(up^s-1)}(n)) \in \{1, 2\},
\]
then the value of \( \nu_p(A_{p,(p-1)(up^s-1)}(n)) \) depends only on the coefficient \( \varepsilon_s \) and the first non-zero coefficient \( \varepsilon_t \) with \( t > s \).

(f) If \( s \geq 2 \) and
\[
\nu_p(A_{p,(p-1)(up^s-1)}(n)) \leq s
\]
for \( n > up^s \), then also
\[
\nu_p(A_{p,(p-1)(up^s-1)}(pn)) = \nu_p(A_{p,(p-1)(up^s-1)}(pn + i)) \text{ for } i = 1, 2, \ldots, p - 1.
\]
In the opposite direction we have the following

**Theorem 17**

Let $k \in \mathbb{N}_+$, $p \in \mathbb{P}_{\geq 3}$ and suppose that $p^2(p - 1) | k$ and $r \in \{1, \ldots, p - 2\}$. Then, there are infinitely many $n \in \mathbb{N}_+$ such that

$$\nu_p(A_{p, k-r}(n)) \geq \nu_p(k).$$
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**Theorem 17**

Let \( k \in \mathbb{N}_+ \), \( p \in \mathbb{P}_{\geq 3} \) and suppose that \( p^2(p-1)|k \) and \( r \in \{1, \ldots, p-2\} \). Then, there are infinitely many \( n \in \mathbb{N}_+ \) such that

\[
\nu_p(A_{p,k-r}(n)) \geq \nu_p(k).
\]

Our computational experiments suggests the following

**Conjecture 7**

Let \( p \in \mathbb{P}_{\geq 3} \), \( u \in \{2, \ldots, p-1\} \) and \( s \in \mathbb{N}_+ \). Then, for \( n \geq up^s \) we have

\[
\nu_p(A_{p,(p-1)(up^s-1)}(n)) \in \{1, 2\}.
\]

Moreover, for each \( n \in \mathbb{N}_+ \) we have the equalities

\[
\nu_p(A_{p,(p-1)(up^s-1)}(pn)) = \nu_p(A_{p,(p-1)(up^s-1)}(pn + i)), \ i = 1, \ldots, p-1.
\]
Let $k \in \mathbb{N}_{\geq 2}$ be given. We say that the sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ is $k$-automatic if and only if the following set

$$K_k(\varepsilon) = \{(\varepsilon_{k^n+j})_{n \in \mathbb{N}} : i \in \mathbb{N} \text{ and } 0 \leq j < k^i\},$$

called the $k$-kernel of $\varepsilon$, is finite.
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In the case of $p = 2$ we know that the sequence $(\nu_2(A_{2,2^s-1}(n)))_{n \in \mathbb{N}}$ is 2-automatic (and it is not eventually periodic). In Theorem 16 we proved that the sequence $(\nu_p(A_{p,k}(n)))_{n \in \mathbb{N}}$ for $k = (p - 1)(p^s - 1)$ with $p \geq 3$, is eventually constant and hence $k$-automatic for any $k$. 
Let $k \in \mathbb{N}_{\geq 2}$ be given. We say that the sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ is $k$-automatic if and only if the following set

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We calculated the first $10^5$ elements of the sequence $(\nu_p(A_{p,(p-1)(u p^s-1)}(n)))_{n \in \mathbb{N}}$ for any $p \in \{3, 5, 7\}$, $s \in \{1, 2\}$ and $u \in \{1, \ldots, p - 1\}$ and were not able to spot any general relations. Our numerical observations lead us to the following

**Question 4**

*For which $p \in \mathbb{P}_{\geq 5}$, $s \in \mathbb{N}$ and $u \in \{2, \ldots, p - 1\}$, the sequence $(\nu_p(A_{p,(p-1)(u p^s-1)}(n)))_{n \in \mathbb{N}}$ is $k$-automatic for some $k \in \mathbb{N}_+$?*
Finally, we formulate the following

**Conjecture 8**

Let $k \in \mathbb{N}_+$, $p \in \mathbb{P}$ and suppose that $k$ is not of the form $(p - 1)(up^s - 1)$ for $s \in \mathbb{N}$ and $u \in \{1, \ldots, p - 1\}$. Then, the sequence $(\nu_p(A_{p,k}(n)))_{n \in \mathbb{N}}$ is unbounded.
Thank you for your attention;-)