Zeros and irreducibility of some classes of special polynomials

Karl Dilcher

Dalhousie Number Theory Seminar
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Part I: Chebyshev-like polynomials

Pafnutiy L’vovich Chebyshev
1821 – 1894
Joint work with

Kenneth B. Stolarsky
University of Illinois, Urbana-Champaign
1. Introduction

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We compute:

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Now consider a slight variant:

\[ V_0(x) = 1, \ V_1(x) = x, \text{ and} \]

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Do we get anything sensible?
Let’s look at a table:
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<table>
<thead>
<tr>
<th>$n$</th>
<th>$V_n(x)$</th>
</tr>
</thead>
<tbody>
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<td>$x^3 - 3x$</td>
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<td>$x^7 - 63x^5 + 49x^3 - 7x$</td>
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<td>$x^8 - 127x^6 + 129x^4 - 31x^2 + 1$</td>
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<td>$x^9 - 255x^7 + 321x^5 - 111x^3 + 9x$</td>
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<tr>
<td>10</td>
<td>$x^{10} - 511x^8 + 769x^6 - 351x^4 + 49x^2 - 1$</td>
</tr>
<tr>
<td>11</td>
<td>$x^{11} - 1023x^9 + 1793x^7 - 1023x^5 + 209x^3 - 11x$</td>
</tr>
<tr>
<td>12</td>
<td>$x^{12} - 2047x^{10} + 4097x^8 - 2815x^6 + 769x^4 - 71x^2 + 1$</td>
</tr>
</tbody>
</table>
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Some properties:

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V_n(x) = \frac{x^{n+2} - T_n(x)}{x^2 - 1}; \quad (1)
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\[ V_n(x) = x^n - \sum_{k=1}^{[n/2]} \binom{n}{2k} (x^2 - 1)^{k-1} x^{n-2k}. \quad (2) \]
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Compare with

\[T_n(x) = x^n + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k}(x^2 - 1)^kx^{n-2k},\]

from which (2) is derived, by way of (1).
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\frac{1 - 2tx}{(1 - tx)(1 - 2tx + t^2)} = \sum_{n=0}^{\infty} V_n(x) t^n. \tag{3}
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2. Irreducibility and Zeros

The Chebyshev polynomial $T_n(x)$ has a well-known factorization over $\mathbb{Q}$ in terms of cyclotomic polynomials. It is irreducible over $\mathbb{Q}$ iff $n = 2^k$, $k = 0, 1, 2, ...$.

How about the $V_n(x)$? Easy to see:

$V_2(x) = (x - 1)(x + 1)$,
$V_4(x) = (x^2 - 3x + 1)(x^2 + 3x + 1)$,

However, all other $V_{2^k}(x)$ and $V_{2^k + 1}(x)$ appear to be irreducible. We can prove a partial result.
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However, all other $V_{2k}(x)$ and $1 + x V_{2k}(x)$ appear to be irreducible.

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Proposition

The following are irreducible over $\mathbb{Q}$:

(a) $V_{2k-2}(x)$ for all $k \geq 3$;

Sketch of Proof: Using the explicit expansion

$V_n(x) = x^n - \lfloor \frac{n}{2} \rfloor - 1 \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \left( \sum_{k=r+1}^{\lfloor \frac{n}{2} \rfloor} (n^2 k^2)(k-1)^r \right) x^{n-2-2r},$

it can be shown that the polynomials in (a) and (b) are 2-Eisenstein. (No other $V_{2k}(x)$ or $1x V_{2k}+1(x)$ is Eisenstein).
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(No other \( V_{2k}(x) \) or \( \frac{1}{x} V_{2k+1}(x) \) is Eisenstein).
Recall: All zeros of $T_n(x)$ lie in the interval $(-1, 1)$. 

Table 2: The largest zeros $r_n$ of $V_n(x)$, $2 \leq n \leq 20$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r_n$</th>
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<tbody>
<tr>
<td>2</td>
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<td>19</td>
<td>1.0170</td>
</tr>
<tr>
<td>20</td>
<td>1.0465</td>
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The zeros of $V_n(x)$ are also all real. However:

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<tr>
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<td>3.8286956</td>
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<tr>
<td>3</td>
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Let $n \geq 2$, and $\pm r_n$ be the largest zeros in absolute value of $V_n(x)$. Then

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Idea of proof: For (a), use \((x^2 - 1)V_n(x) = x^{n+2} - T_n(x)\).
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Idea of proof: For (a), use $(x^2 - 1)V_n(x) = x^{n+2} - T_n(x)$. Consider graph of $y = T_n(x)$; count intersections with $y = x^{n+2}$. 

[Graph of polynomial and its relationship with $x^{n+2}$]
Let \( n \geq 2 \), and \( \pm r_n \) be the largest zeros in absolute value of \( V_n(x) \). Then

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(b): Evaluate \( V_n(x) \) at the two boundary points of the interval.

\[ T_{20}(x) \]
3. A Related Polynomial

The Chebyshev polynomials $T_n(x)$ satisfy the (2×2 Hankel determinant) identity

$$T_{n+1}(x) - T_n(x)T_{n+2}(x) = 1 - x^2 \quad (n \geq 0).$$

How about the analogue for \{\{V_n(x)\}\}?

Define $W_n(x) := V_{n+1}(x) - V_n(x)V_{n+2}(x) \quad (n \geq 0)$. We'll see: These polynomials have some interesting properties.
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Recurrence: \( W_0(x) = 1, \ W_1(x) = x^2 + 1, \) and for \( n \geq 1, \)

\[ W_{n+1}(x) = x^2 (2W_n(x) - W_{n-1}(x)) + 1. \]
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Generating function:

\[ \frac{1 - tx^2 + t^2x^2}{(1 - t)(1 - 2tx^2 + t^2x^2)} = \sum_{n=0}^{\infty} W_n(x)t^n. \]
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Do we get anything sensible if we cut the $W_n(x)$ into two halves?
Define the lower and upper parts, respectively, of $W_n(x)$ by

$$W_n^\ell(x) := \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} x^{2j},$$

$$W_n^u(x) := \frac{1}{x^{n+2}} \left( W_n(x) - W_n^\ell(x) \right).$$
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Easy to establish generating functions for both, and with these we get

$$W_n^u(x) = 2 \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} U_{n-2-2k}(x)$$

where the $U_n(x)$ are the Chebyshev polynomials of the second kind, which can be defined by the generating function

$$\frac{1}{1 - 2tx + t^2} = \sum_{n=0}^\infty U_n(x)t^n.$$
Using known identities:

\[
W_{2k}^u(x) = \frac{1 - T_{2k}(x)}{1 - x^2} = 2U_{k-1}(x)^2,
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This, together with the definition of the \( W_n^\ell(z) \), gives

**Proposition**

*For all \( n \geq 1 \), the zeros

(a) of \( W_n^\ell(z) \) lie on the unit circle;

(b) of \( W_n^u(z) \) lie in the open interval \( (-1, 1) \).*
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\[ W_{2k}^u(x) = \frac{1 - T_{2k}(x)}{1 - x^2} = 2U_{k-1}(x)^2, \]
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What can we say about the zeros of \( W_n(z) \) as a whole?
Plot of the zeros of \( W_{50}(z) \) (degree 100):
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Do they lie on (or near) an identifiable curve?
Proposition

The zeros of \( W_n(z) \), as \( n \to \infty \), lie arbitrarily close to the curve

\[
3r^8 - 8r^6 \cos(2\theta) + 6r^4 - 1 = 0, \quad z = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi. \tag{4}
\]

Furthermore, they all lie outside the closed region defined by this curve.
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Figure: The zeros of $W_{50}(z)$ and the curve (4).
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"You want proof? I'll give you proof!"
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- A chain of tricky estimates.
An older result of a similar flavour:

Let $L_p(x)$, $U_p(x)$ be the lower and upper sections of an even-degree polynomial $p(x)$.

Proposition (D. & Stolarsky, 1992)

There is a sequence of polynomials $\{Q_n(x)\}$ such that

(a) the zeros of $Q_n(x)$ lie on the oval $|x(x-1)| = \frac{1}{2}$;

(b) the zeros of $LQ_n(x)$ lie on the circle of radius $\frac{1}{\sqrt{2}}$ centered at the origin;

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Remarks: (i) The centers of the circles in (b), (c) are the foci of the oval (an oval of Cassini) in (a).

(ii) The polynomials can be given explicitly and are also related to Chebyshev polynomials.
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Part II:

Zeros and irreducibility of gcd-polynomials
Joint work with

Sinai Robins
University of São Paulo, Brazil
1. Introduction

Some classes of polynomials with special number theoretic sequences as coefficients:

Fekete polynomials:

\[ f_p(z) := p^{-1} \sum_{j=0}^{p-1} (j^p)_p z^j \]  

where \((a_p)\) is the Legendre symbol.

Conrey, Granville, Poonen, and Soundararajan (2000) showed:

For each \(p\), at least half of the zeros of \(f_p(z)\) lie on the unit circle.

Deep connections with the distribution of primes.
Some classes of polynomials with special number theoretic sequences as coefficients:

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Deep connections with the distribution of primes.
2. Ramanujan polynomials:

\[ R_{2k+1}(z) := \sum_{j=0}^{k+1} \left( \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!} \right) z^{2j}, \]

where \( B_n \) is the \( n \)th Bernoulli number.
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Later extended by other authors to similar polynomials (Lalín & Smyth, 2013; Berndt & Straub, 2017).
3. Dedekind polynomials:

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with $\left( (x) \right)$ denoting the “sawtooth function”

$$\left( (x) \right) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2}, & \text{otherwise.} \end{cases}$$
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\[ ((x)) = \begin{cases} 
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 x - [x] - \frac{1}{2}, & \text{otherwise.} 
\end{cases} \]

Observation:
For each \( k \), most of the zeros of \( p_k(z) \) lies on the unit circle.
3. Dedekind polynomials:

\[ p_k(z) := \sum_{j=0}^{k-1} s(j, k)z^j, \]

where \( s(d, c) \) is the Dedekind sum defined by

\[ s(d, c) = \sum_{j=1}^{c} \left( \left( \frac{j}{c} \right) \right) \left( \left( \frac{dj}{c} \right) \right), \]

with \( \left( \left( x \right) \right) \) denoting the "sawtooth function"

\[ \left( \left( x \right) \right) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x - \lfloor x \rfloor - \frac{1}{2}, & \text{otherwise}. \end{cases} \]

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In an effort to prove this, we were led to studying the following class of polynomials.
What can we say about the polynomials
\[ \sum_{j=0}^{n} \gcd(n, j)z^j? \]

It turns out: A more general class has basically the same properties.

For \( k \geq 0 \) and \( n \geq 1 \), let \( g(k)n(z) := \sum_{j=0}^{n} \gcd(n, j)kz^j \).

For \( k = 0 \), obviously \( g(0)n(z) = z^n+1-1z^{-1} \), so all the zeros are roots of unity and thus lie on the unit circle. For \( n = p-1 \) (\( p \) a prime), these are cyclotomic polynomials; hence irreducible.
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For $n = p - 1$ ($p$ a prime), these are cyclotomic polynomials; hence irreducible.
From now on: Disregard the case $k = 0$. However, we will see: $g(k)\eta(z)$ for $k \geq 1$ have properties similar to the case $k = 0$. Theorem For all $k \geq 1$ and all $n \geq 1$, all the zeros of $g(k)\eta(z)$ lie on the unit circle and have uniform angular distribution. Idea of proof: Consider $g(k)\eta(e^{2\pi ix})$ and show it has $n$ real zeros for $0 < x < 1$. Karl Dilcher Zeros and irreducibility of some classes of special polynomials
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*For all $k \geq 1$ and all $n \geq 1$, all the zeros of $g_n^{(k)}(z)$ lie on the unit circle and have uniform angular distribution.*
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**Theorem**

*For all $k \geq 1$ and all $n \geq 1$, all the zeros of $g_n^{(k)}(z)$ lie on the unit circle and have uniform angular distribution.*

Idea of proof: Consider

$$g_n^{(k)}(e^{2\pi ix})$$

and show it has \textit{n real} zeros for $0 < x < 1$. 
Since \( \gcd(j, n) = \gcd(n - j, n) \) for \( 0 \leq j \leq n \), the \( g_n^{(k)}(z) \) are self-inversive (or reciprocal):

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g_n^{(k)}(z) = z^n g_n^{(k)}\left(\frac{1}{z}\right).
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\[ S^{(k)}(m, n) := \sum_{j=1}^{n} \gcd(j, n)^k e^{2\pi ijm/n}. \]
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Thus, if we can show

\[ S^{(k)}(m, n) > 0, \]  \hspace{2cm} (5)

then for fixed \( k \) and \( n \), \( h^{(k)}_n \left( \frac{m}{n} \right) \) is alternating positive and negative.
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• arithmetic, especially multiplicative, functions in general;
• the gcd and its powers as special cases.

For instance:

Theorem (L. Tóth, 2011)
For all \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \),
\[ S^{(1)}(m, n) = \sum_{d \mid \gcd(m, n)} d \varphi\left( \frac{n}{d} \right). \]
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We need a generalization of Euler’s $\varphi$-function.

**Definition**

Jordan’s totient function is defined by

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right),$$

or equivalently as the number of different sets of $k$ (equal or distinct) positive integers $\leq n$ whose gcd is relatively prime to $n$.

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W. Schramm (2008) showed;

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This can be extended:
Proposition

For all $k, n \in \mathbb{N}$ and all $m \in \mathbb{Z}$ we have

$$S^{(k)}(m, n) = \sum_{d \mid \gcd(m, n)} dJ_k\left(\frac{n}{d}\right).$$

In particular, $S^{(k)}(m, n)$ is always a positive integer.
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Compare with Tóth’s result:

\[
S^{(1)}(m, n) = \sum_{d \mid \gcd(m,n)} d\varphi\left(\frac{n}{d}\right).
\]
Proof of Proposition. Using

\[ \gcd(j, n)^k = \sum_{d \mid \gcd(j, n)} J_k(d), \]

we have

\[ S^{(k)}(m, n) = \sum_{j=1}^{n} \sum_{\ell \mid \gcd(n, j)} J_k(\ell) e^{2\pi ijm/n} \]

\[ = \sum_{\ell \mid n} J_k(\ell) \sum_{j=1}^{n/\ell} e^{2\pi ijm/(n/\ell)}. \]
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  \item 0 otherwise.
\end{itemize}

Hence, setting $d = n/\ell$, we get the desired identity.
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Set \( m = n \); then

**Corollary**

*For all \( k, n \in \mathbb{N} \) we have*

\[ \sum_{d \mid n} dJ_k \left( \frac{n}{d} \right) = \sum_{j=1}^{n} \gcd(j, n)^k. \]
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This was published by K. Alladi (1975) when he was 19 years old, and with a different goal in mind.
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**Theorem**

For \( \alpha, k \in \mathbb{N} \) and odd primes \( p \),
\[ g_{2\alpha}^{(k)}(z) \quad \text{and} \quad \frac{g_p^{(k)}(z)}{z + 1} \]
are irreducible over \( \mathbb{Q} \).
**Proof.** (Sketch).

**Part 1:** We begin with the smallest cases:

\[ g_2^{(k)}(z) = 2^k + z + 2^k z^2, \quad \frac{1}{z+1}g_3^{(k)}(z) = 3^k + (1 - 3^k)z + 3^k z^2. \]
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But none of the polynomials above are of this form.
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For the remaining cases, let \( p \geq 2 \) be any prime, and \( \alpha, k \in \mathbb{N} \).
Set

\[ g_n^{(k)}(z) = \begin{cases} g_n^{(k)}(z) & \text{when } n \text{ is even}, \\ \frac{1}{z+1} g_n^{(k)}(z) & \text{when } n \text{ is odd}. \end{cases} \]
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So we can write, for any $n \geq 4$,

$$g_n^{(k)}(z) = (a_1 + b_1 z + \ldots)(a_2 + b_2 z + \ldots)\ldots(a_r + b_r z + \ldots)$$

$$= a_1 a_2 \ldots a_r + a_1 a_2 \ldots a_r \left( \sum_{j=1}^{r} \frac{b_j}{a_j} \right) z + \ldots$$
On the other hand, it is clear from the definition that

\[
\overline{g}_{p^\alpha}^{(k)}(z) = \begin{cases}
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Equating coefficients, we therefore have

\[ a_1 a_2 \ldots a_r = p^{\alpha k}, \quad (6) \]

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On the other hand, it is clear from the definition that

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- By (6): the \( a_j \) can only be powers of \( p \);
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(otherwise \( p \) would divide LHS of (7) — contradiction.)
This means: at least one of the $r$ irreducible factors (which are self-inversive) is monic, with all its zeros on the unit circle.
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Leopold Kronecker 1823 – 1891
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Proof requires a detailed analysis using resultants of polynomials.

We skip this.
Thank you
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We denote the resultant of \( f \) and \( g \) by

\[ \text{Res}(f, g) \]

if there is no ambiguity as to the variable \( z \).
Suppose that the zeros of $f$ and $g$ are $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$, respectively. Then the most important property is

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\text{Res}(f, g) = a_n^m b_m^n \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j),
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an alternative definition.

Some consequences:

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\text{Res}(f, g) = a_n^m b_m^n \prod_{i=1}^{m} g(\alpha_i),
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The first identity shows that $\text{Res}(f, g) = 0$ iff $f$ and $g$ have a factor in common.
Important for us:

**Lemma (Apostol (1970))**

*For* $m > n > 1$ *we have*

$$\text{Res}(\Phi_m(z), \Phi_n(z)) = \begin{cases} p^{\varphi(n)} & \text{if } \frac{m}{n} \text{ is a power of a prime } p, \\ 1 & \text{otherwise}. \end{cases}$$
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With this we will prove

**Lemma**

Let $p$ be any prime and $\alpha, k$ be positive integers. Then

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unless \( n = 2^j d \) for some nonzero \( j \) and \( d > 1 \) where \( d \mid p^\alpha + 1 \) (\( j \) may be positive or negative).
(b) On the other hand,

\[ g_{p^{\alpha}}^{(k)}(z) \equiv (z + \cdots + z^{p-1}) + (z^{p+1} + \cdots + z^{2p-1}) \]
\[ + \cdots + (z^{p^{\alpha}-p+1} + \cdots + z^{p^{\alpha}-1}) \pmod{p} \]
\[ = z \left( 1 + z + \cdots + z^{p-2} \right) \left( 1 + z^{p} + \cdots + z^{(p^{\alpha}-1)p} \right) \]
\[ = z \cdot \frac{z^{p-1} - 1}{z - 1} \cdot \frac{z^{p^{\alpha}} - 1}{z^{p} - 1} \]
\[ = z \prod_{d|p-1, d \neq 1} \Phi_d(z) \prod_{j=2}^{\alpha} \Phi_{p^j}(z). \]
By properties of resultants,

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Combining the conditions:

The above congruences (mod 2) and (mod \( p \)) fail simultaneously only if

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This completes the proof of the resultant lemma, and thus of the irreducibility theorem.
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Still, we propose

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Verified by computation for all \( n \leq 1000 \) and \( 1 \leq k \leq 10 \).
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2. Our results give a large supply of algebraic numbers on the unit circle that are not roots of unity.
Thank you