Valuative Capacity of some compact subsets of $\mathbb{Z}_p$

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A $p$-ordering of an infinite set, $S \subseteq \mathbb{Z}_p$, is a sequence in $S$ such that for $\forall n > 0$, $a_n$ minimizes

$$v_p((x - a_{n-1}) \ldots (x - a_0))$$
Background: \( p \)-orderings, \( p \)-sequences

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\]

cf: A \( \rho \)-\textbf{ordering} of \( S \), a (compact) subset of an ultrametric space \((M, \rho)\), is a sequence in \( S \) such that \( \forall n > 0 \), \( a_n \) maximizes

\[
\prod_{i=0}^{n-1} \rho(x, a_i)
\]
Background: $p$-orderings, $p$-sequences

The $p$-sequence of $S$ is the sequence whose $0^{th}$-term is 1 and whose $n^{th}$ term, for $n > 0$, is

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The **valuative capacity** of an infinite set, $S \subseteq \mathbb{Z}_p$, is

$$L_p(S) := \lim_{n \to \infty} \frac{w_S(n, p)}{n}$$

where $w_S(n, p)$ is the $p$–sequence of $S$. 

\[nb:\text{this is the Robin's constant and can be found via the equilibrium measure:}\]

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Let $A = \{0, 1, \ldots, d - 1\}$ be a finite alphabet and $A^\mathbb{N}$ be the collection of infinite sequences with values in $A$.

Let $p \geq d$ be a prime number and let $\phi$ be the canonical embedding of $A^\mathbb{N}$ into $\mathbb{Z}_p$ via the following continuous map:

$$
\phi : A^\mathbb{N} \rightarrow \mathbb{Z}_p \text{ by } (x_n)_{n \geq 0} \mapsto \sum_{k=0}^{\infty} x_k p^k
$$
Lemma

Let $w_1, w_2, \ldots, w_s$ be $s \geq 2$ words with the same length $l$ such that all the first letters are distinct. Let $X \subset A^\mathbb{N}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \ldots, w_s$. Then the set $E := \phi(X) \subset \mathbb{Z}_p$ satisfies:

$$E = \bigcup_{i=1}^s x_i + p^l E,$$

with $x_i = \phi(w_i0^n)$

It is a regular compact set and its valuative capacity is

$$L_p(E) = \frac{l}{s - 1}$$

Notice that this provides examples of sets with empty interiors but with positive capacities.
An example:

\[ w_1 = 0, \; w_2 = 2, \; A = \{0, 1, 2\}, \; p = d = 3 \]

Then \( \{x_n\}_{n \geq 0} \in X \) if each term in \( \{x_n\}_{n \geq 0} \) is either 0 or 2. We have

\[
E = 0 + 3E \cup 2 + 3E \quad \text{and} \quad L_p(E) = \frac{1}{2 - 1} = 1
\]
Digression: projective $k$-space

Let $k$ be a field that is complete with respect to a non-archimedean valuation.

**Definition**
The **projective line over** $k$, denoted $\mathbb{P}^1(k)$, is the space whose points are lines $l$ in $k^2$ that intersect $(0,0)$.

**Proposition**
Let $\psi : k \to \mathbb{P}^1(k)$ be the map given by $\psi(\lambda_0) = [1, \lambda_0]$, where $[1, \lambda_0]$ is the line in $k^2$, $\{\lambda(1, \lambda_0); \lambda \in k^*\}$. Then the image of $\psi$ is $\mathbb{P}^1(k) \setminus [0, 1]$ and is isomorphic to $k$, so that $k$ is identified with projective space minus a distinguished point, $[0, 1]$, which is denoted by $\infty$. 
Definition
We denote by $GL(2, k)$ the set of invertible $2 \times 2$ matrices over $k$. A fractional linear automorphism, $\phi$, of $\mathbb{P}^1(k)$ is a map defined by $z \mapsto \frac{az+b}{cz+d}$ for some $(a \ b \ c \ d) \in GL(2, k)$. The set of fractional linear automorphisms of $\mathbb{P}^1(k)$ is denoted $PGL(2, k)$.

Note that $PGL(2, k) = GL(2, k)/\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \lambda \in k^*\}$. In homogeneous coordinates, we can represent the action of $\phi$ by $[x_0, x_1] \mapsto [cx_1 + dx_0, ax_1 + bx_0]$. 

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**Definition**
Suppose $\Gamma$ is a subgroup of $\text{PGL}(2, k)$. A point $p \in \mathbb{P}^1(k)$ is a **limit point of** $\Gamma$, if there exists a point $q$ in $\mathbb{P}^1(k)$ and a sequence $\{\gamma_n\}_{n \geq 1}$ in $\Gamma$ such that $\lim_{n \to \infty} \gamma_n(q) = p$. 
Let $x_1, x_2, \ldots, x_s$ be $s \geq 2$ points in $\mathbb{Z}_p$ such that $|x_i - x_j|_p = 1, \forall i, j \in 1, \ldots, s$. Suppose also that there exists an $l \in \mathbb{N}$ such that $\forall i$,

$$x_i = \sum_{i=0}^{\infty} a_i p^i = \sum_{i=0}^{l} a_i p^i$$
Let $\gamma_i$ be the fractional linear automorphism of $\mathbb{P}^1(\mathbb{Q}_p)$ given by $\begin{pmatrix} p^l & x_i \\ 0 & 1 \end{pmatrix}$ and let $\Gamma$ be the subgroup of $PGL(2, \mathbb{Q}_p)$ generated by the $\gamma_i$.

Then $\Gamma$ has a subgroup $H$ such that the limit set $\mathcal{L}$ of $H$ has the property that $Z = \psi^{-1}(\mathcal{L})$ is equal to $\phi(X)$ in the original lemma. In particular $Z$ is a regular, compact subset of $\mathbb{Z}_p$ satisfying

$$Z = \bigcup_{i=1}^s x_i + p^l Z = \bigcup_{i=1}^s B_{\frac{1}{p^l}}(x_i)$$

and with vaulative capacity

$$L_p(Z) = \frac{l}{s - 1}$$
Fares and Petite, Lemma 5.1, rephrased

Sketch of proof:

- We have to show that the set $Z$ above is equal to $E = \phi(X)$ in the original lemma.
- That the $w_i$ correspond to the $x_i$ is not hard to see.
- What is the limit set of $\Gamma$?
Limit set of $\Gamma$

Let $\gamma \in \Gamma$.

- If $\gamma$ is a product of the generators $\gamma_i$, then $\gamma$ is represented by a matrix of the form: $\begin{pmatrix} p^l m & z \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and $z$ is an element of $\mathbb{Z}_p$ whose coefficient vector is a concatenation of the coefficient vectors of the $x_i$ (for $0 \leq i \leq ml$ and $0$ for $i > ml$).

- For example,

$$\begin{pmatrix} p^l x_i \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} p^l x_j \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} p^l x_k \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p^{3l} p^{2l} x_k + p^l x_j + x_i \\ 0 \\ 1 \end{pmatrix}$$

- The action of this map is given by

$$[a_0, a_1] \mapsto [a_0, p^{lm} a_1 + z a_0] \sim [1, p^{lm} \frac{a_1}{a_0} + z]$$
Let $\gamma \in \Gamma$.

- If $\gamma$ is a product of the inverses of the generators $\gamma_i^{-1}$, then $\gamma$ is represented by a matrix of the form: $\begin{pmatrix} p^{-lm} & -p^{-l}z^{-1} \\ 0 & 1 \end{pmatrix}$, where $m \in \mathbb{N}$ and $z$ is as above.

- For example,

$$ \begin{pmatrix} p^{-l} & -p^{-l}x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{-l} & -p^{-l}x_k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-3l} & -p^{-3l}x_k - p^{-2l}x_j - p^{-l}x_i \\ 0 & 1 \end{pmatrix} $$

- The action of this map is given by

$$ [a_0, a_1] \mapsto [a_0, p^{-lm}a_1 - p^{-l}z^{-1}a_0] \sim [1, p^{-l}(p^{-m}\frac{a_1}{a_0} - z^{-1})] $$
Limit set of $\Gamma$

Let $\gamma \in \Gamma$.

- If $\gamma$ is of the form $\gamma_{\gamma}^{-1}\gamma_i$, for $i \neq j$, then $\gamma$ is represented by a matrix of the form: \[
\begin{pmatrix}
1 & p^{-l}(x_i-x_j) \\
0 & 1
\end{pmatrix}
\]

- The action of this map is given by

\[
[a_0, a_1] \mapsto [a_0, a_1 + p^{-l}(x_i-x_j)a_0] \sim [1, \frac{a_1}{a_0} + p^{-l}(x_i-x_j)]
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Let $\gamma \in \Gamma$.

- If $\gamma$ is of the form $\gamma_j \gamma_i^{-1}$, for $i \neq j$, then $\gamma$ is represented by a matrix of the form: 
  $\begin{pmatrix}
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  [a_0, a_1] \mapsto [a_0, a_1 + (x_i - x_j)a_0] \sim [1, \frac{a_1}{a_0} + (x_i - x_j)]
  \]

- We quotient the group $\Gamma$ by the group generated by the translations to obtain $H$. 
In fact, all of the translations commute with each other, so we can quotient by the entire translation subgroup, i.e., the subgroup generated by \( \{ \gamma_i \gamma_j^{-1}, \gamma_i^{-1} \gamma_j; \forall i, j \in 1, \ldots, s \} \). The resulting quotient group is discontinuous, finitely generated, and every element (\( \neq \text{id} \)) is hyperbolic, i.e., it is a Schottky group.
In fact, all of the translations commute with each other, so we can quotient by the entire translation subgroup, i.e., the subgroup generated by \( \{ \gamma_i \gamma_j^{-1}, \gamma_i^{-1} \gamma_j; \forall i, j \in 1, \ldots, s \} \)

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Consider the following:

\[
S \subseteq \mathbb{Z}_p \quad \rightarrow \quad \mathbb{Q}_p
\]

\[
\downarrow \quad \quad \quad \downarrow
\]

\[
\mathbb{P}(\mathbb{Z}_p) \quad \quad \mathbb{P}(\mathbb{Q}_p)
\]
references

Youssef Fares and Samuel Petite, The valuative capacity of subshifts of finite type.

Keith Johnson, P-orderings and Fekete sets