The Regularity of Singularity

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Take an infinite field $\mathcal{K}$ and $S = \mathcal{K}[x_1, \dots , x_n]$.

Question. Suppose that $\operatorname{char} \mathcal{K} = 0$. Given a set $X \subset \mathcal{K}^n$ find the codimension in the space of polynomials in S of degrees $\leq m$ that are singular to order $k-1$ at each point of X. (This may be extended naturally to any infinite field.)

We are particularly interested in an algebraic variety X: a set defined by polynomial equations of S. This includes any finite collection of points in $\mathcal{K}^n$. Further, to gain insight on the question for any variety X one may begin by examining finite sets; namely, those given by slicing X by planes. Thus we may begin with a focus on a set $\Gamma = \{p_1, \dots , p_d\} \subset \mathcal{K}^n$.

One may recall (or not!) how we examined this question for a *general* collection of d points: "Γ is chosen at random". In that case, the "fewest" polynomials of given degree m are singular (and are zero) on Γ, with respect to d, n, and m. We shall discuss here the problem for an arbitrary collection, with the broad conclusion: loosely speaking, if there are "many" polynomials that are singular on Γ for each degree and order of singularity, then there must be "many" polynomials that are zero on Γ as well. Thus we must quantify this statement in order to attach a meaning.

We define, then, the **regularity** of an ideal and we show a comparison between the regularity of the ideal of the polynomials that are zero on X and and that of the ideal defined by those polynomials that are singular to given order on X.

The methods used here are purely algebraic, thus one may feel free to ignore any geometric interpretations presented!