The Regularity of Singularity, II

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Let $\mathcal{K}$ be an infinite field, $\mathbb{P}^n = \mathbb{P}^n_{\mathcal{K}}$, and $S = \mathcal{K}[X_0, \dots, X_n]$ as the graded homogeneous ring of polynomials.
Previously we examined the following issues.

Question. Suppose that $\operatorname{char} \mathcal{K} = 0$. Given a set $X \subset \mathcal{K}^n$ take the ideal $I(X)^{<k>}$ of polynomials that are singular to order $k-1$ at each point of X (including $I(X)^{<1>} = I(X)$). Find the codimension of $I(X)^{<k>}$ in S. (This may be extended naturally to any infinite field.)

We are particularly interested in an algebraic variety X: a set defined by polynomial equations of S. This includes any finite collection of points in $\mathcal{K}^n$. Further, to gain insight on the question for any variety X one may begin by examining finite sets; namely, those given by slicing X by planes. Thus we may begin with a focus on a set $\Gamma = \{p_1, \dots, p_d\} \subset \mathcal{K}^n$.

One may recall (or not!) how we examined this question for a *general* collection of d points: "Γ is chosen at random". In that case, the "fewest" polynomials of given degree m are singular (and are zero) on Γ, with respect to d, n, and m. We shall discuss here the problem for an arbitrary collection, with the broad conclusion: loosely speaking, if there are "many" polynomials that are singular on Γ for each degree and order of singularity, then there must be "many" polynomials that are zero on Γ as well. Thus we must quantify this statement in order to attach a meaning.

We considered the ideal $I(X)^k = < \{F_1 F_2 \dots F_k : F_1, \dots, F_k \subset I(X)\} >$ and its relation to $I(X)^{<k>}$; further, the ideal I^k is defined for any ideal I.

We defined the notion of *regularity* of an ideal, $\operatorname{reg} I$, determined by its minimal free resolution. The regularity tells when the ideal starts to "act predictably". We shall also refer to the modules of syzygies of I, the saturation of an ideal, and the Krull dimension of a module.
We show:

Theorem 1 *Suppose that $I \subset S$ is an ideal with Krull dimension $\dim S/I \leq 1$. Suppose that I is generated in degree at most m, its first syzygy module is generated in degree at most $s+1$, and that I is not principal. Then for each $k \geq 2$ we have $\operatorname{reg} I^k \leq (k-2)m + s + \operatorname{reg} I$.*

We shall comment on the extension to the case of $\dim S/I = 2$.