Counting Polynomials with Higher-Order Singularities
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Take an infinite field $\mathcal{K}$ and the ring of polynomials $S = \mathcal{K}[x_1, ..., x_n]$ on $\mathcal{K}^n$.
Question. Given $p_1, \ldots, p_d \in \mathcal{K}^n$, what is the dimension of the space of polynomials of degrees at most m that are singular to order k at each point? What is the maximal possible dimension (based on the choice of the d points)?

The answer is well-known in the case of polynomials in one variable ($n = 1$), according to the classical theorem of Lagrange-Hermite. In higher dimensions, let us deal with the general case ("points chosen randomly"). For $n = 2$ the natural conjecture have been made (Segre, Harbourne, Hirschowitz) with some verifications when $k \leq 20$. As well, the question has been solved for $k = 1$ in every dimension (Alexander-Hirschowitz, Chandler). Roughly speaking, one sees here a natural extension of the Lagrange-Hermite Theorem up to sparse exceptions.

For $n \geq 3$ and $k \geq 2$ however such generalisations become naïve, particularly in low degree. Instead, we have seen how algebraic conjectures of Iarrobino should apply from geometric considerations. We have shown inductive strategies on verifying these conjectures. Here we shall illustrate these methods to prove the following:

Theorem. Let $\Gamma \subset K^n$ be a general collection of points. Take I as the space of polynomials singular to order two on Γ, S_m as the space of polynomials of degree at most m, and $I_m = I \cap S_m$. Then for $m \geq 8$ we have, as hoped,

$$\dim S_m/I_m = \binom{n+2}{2} d.$$

However, in lower degrees (certainly $m \leq 4$) we cannot expect such an equality according to geometric obstructions. This is relevant, as the argument proceeds by induction on dimension and degree so that we must understand such obstructions.