Introduction to Zonal Polynomials

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Dalhousie University Number Theory Seminar

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Hypergeometric function and Pochhammer symbol

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- for $p = (p_1, \ldots, p_l) \in \mathcal{P}_n$, $(a)_p = \prod_{i=1}^{l} \left(a - \frac{i-1}{2}\right)_{p_i}$;
- $\mathcal{C}_p(Y)$ is (C*-normalization of*) zonal polynomial, which is homogeneous, symmetric, polynomial of degree $n = |p|$, in the eigenvalues of Y.

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- An important fact

$$\sum_{p \in \mathcal{P}_n} \mathcal{C}_p(Y) = (\mathrm{tr}\, Y)^n = (y_1 + \cdots + y_m)^n \,.$$

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$$ {}_0F_0\left(\;:\; Y\right) = \sum_{n=0}^{\infty} \sum_{p \in \mathcal{P}_n} \cdot \frac{\mathcal{C}_p(Y)}{n!} = \sum_{n=0}^{\infty} \frac{(\operatorname{tr} Y)^n}{n!} = e^{\operatorname{tr} Y} $$

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$$ {}_1F_0\left({}^a \; : z \right) = (1 - z)^{-a} $$

$$ {}_1F_0\left({}^a \; : Y \right) = \det(I - A)^{-a} $$

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  $$u_r(x_1, \ldots, x_m) := \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

  Then, for $p = (p_1, \ldots, p_l) \in \mathcal{P}_n$

  $$\mathcal{U}_p := u_1^{p_1 - p_2} u_2^{p_2 - p_3} \cdots u_{l-1}^{p_{l-1} - p_l} u_l^{p_l(-0)},$$

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  - $\mathcal{U} := (\mathcal{U}_{(n)}, \mathcal{U}_{(n-1,1)}, \ldots, \mathcal{U}_{(1,1,\ldots,1)})^T$ forms a basis of V_n.

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- Let $X_{\nu \times m}$ be such that each row is independently drawn from an m-variate normal distribution,

$$\left(x_i^1, \ldots, x_i^m\right) \sim \mathcal{N}_m (0, V) \Rightarrow S = X^T X \sim W_m (V, \nu)$$

  and ν is called the degree of freedom.

Computation

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$$\mathcal{M}_\lambda(y_1, \ldots, y_m) = \sum_{\substack{i_1, \ldots, i_l \\ \text{distinct terms}}} y_{i_1}^{\lambda_1} \cdots y_{i_l}^{\lambda_l} = y_1^{\lambda_1} \cdots y_l^{\lambda_l} + \text{symmetric terms.}$$

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2. $$\mathcal{M}_{(2)}(Y) = y_1^2 + \cdots + y_m^2;$$
3. $$\mathcal{M}_{(1,1)}(Y) = \sum_{i<j} y_i y_j;$$
4. $$\mathcal{M}_{(2,1)}(Y) = \sum_{i,j} y_i^2 y_j.$$

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For $p = (p_1, \ldots, p_\ell)$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$,

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for some constants $c_{p,\lambda}$

$$c_{p,\lambda} = \sum_{\lambda < \mu \leq p} \frac{(\lambda_i + t) - (\lambda_j - t)}{\rho_p - \rho_\lambda} c_{p,\mu}.$$

Here,

$$\rho_p := \sum_{j=1}^{\ell} p_i (p_i - j)$$

and for $\lambda = (\lambda_1, \ldots, \lambda_l)$, the sum is over all $\mu = (\lambda_1, \ldots, \lambda_i + t, \ldots, \lambda_j - t, \ldots, \lambda_l)$ for $t = 1, \ldots, \lambda_j$ such that by rearranging tuple μ in a descending order, it lies as $\lambda < \mu \leq p$.

Computation

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- $n = 5$

$p\backslash\lambda$	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
(5)	1	$\frac{5}{9}$	$\frac{10}{21}$	$\frac{20}{63}$	$\frac{2}{7}$	$\frac{4}{21}$	$\frac{8}{63}$
$(4,1)$	0	$\frac{40}{9}$	$\frac{8}{3}$	$\frac{46}{9}$	4	$\frac{14}{3}$	$\frac{40}{9}$
$(3,2)$	0	0	$\frac{48}{7}$	$\frac{32}{7}$	$\frac{176}{21}$	$\frac{64}{7}$	$\frac{80}{7}$
$(3,1,1)$	0	0	0	10	$\frac{20}{3}$	$\frac{130}{7}$	$\frac{200}{7}$
$(2,2,1)$	0	0	0	0	$\frac{32}{3}$	16	32
$(2,1,1,1)$	0	0	0	0	0	$\frac{80}{7}$	$\frac{800}{21}$
$(1,1,1,1,1)$	0	0	0	0	0	0	$\frac{16}{3}$

Computation

- $n = 5$

$p \backslash \lambda$	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
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- $$\mathcal{C}_{(1,1)}(a,b,c) = \frac{4}{3}(ab+bc+ac)$$
$$\mathcal{C}_{(2)}(a,b,c) = a^2+b^2+c^2+\frac{2}{3}(ab+bc+ac)$$

Computation

Computation

Applications Places System

IPython: Math/RISC&RICAM

File Edit View Search Terminal Help

```
ljiu@ljiu:~/Desktop/Now/Math/RISC&RICAM$ sage
┌────────────────────────────────────────────────────────────────────┐
│ SageMath version 7.4, Release Date: 2016-10-18                     │
│ Type "notebook()" for the browser-based notebook interface.        │
│ Type "help()" for help.                                            │
└────────────────────────────────────────────────────────────────────┘
sage: load('Zonal.sage')
sage: var('a','b','c')
(a, b, c)
sage: MZonal([2,1],[a,b,c])
a^2*b + a*b^2 + a^2*c + b^2*c + a*c^2 + b*c^2
sage: CZonal([2,1],[a,b,c])
12/5*a^2*b + 12/5*a*b^2 + 12/5*a^2*c + 18/5*a*b*c + 12/5*b^2*c + 12/5*a*c^2 + 12/5*b*c^2
sage: coeffi([2,1],[1,1,1])
18/5
sage: 
```

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On a Riemannian manifold (M, g), the Laplace-Beltrami operator on $f \in C^\infty(M)$ is given by

$$\Delta f := (\mathrm{div} \bullet \mathrm{grad})\, f = \frac{1}{\sqrt{G}} \partial_k \left(g^{ik} \sqrt{G} \partial_i f \right).$$

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$$\Delta = \sum_{i=1}^{m} \left(y_i^2 \frac{\partial^2}{\partial y_i^2} - \underline{\frac{m-3}{2} y_i \frac{\partial}{\partial y_i}} + \sum_{j=1, j \neq i}^{n} \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i} \right).$$

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Zonal polynomial $\mathcal{C}_p(y_1, \ldots, y_m)$ are eigenfunctions of Δ_Y, defined by

$$\Delta_Y := \sum_{i=1}^{m} \left(y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{j=1, j\neq i}^{n} \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i} \right).$$

In particular

$$\Delta_Y \mathcal{C}_p(Y) = (\rho_p + m(l-1))\, \mathcal{C}_\lambda(Y), \text{ where } \rho_p := \sum_{i=1}^{l} p_i (p_i - 1).$$

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$$\begin{aligned} g \in \mathrm{GL}(m) : V_n &\to V_n \\ \varphi(Y) &\mapsto \varphi\left(g^{-1} Y \left(g^{-1}\right)^T\right) \end{aligned}$$

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Now, note that $(\mathrm{tr}\, Y)^n \in V_n$. The projection

$$(\mathrm{tr}\, Y)^n \Big|_{V_p} = \mathcal{C}_p(Y).$$

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First fix some notation:

- R is a finite root system in a real vector space V.
- R^+ is a choice of positive roots, to which corresponds a positive Weyl chamber.
- W is the Weyl group of R.
- Q is the root lattice of R (the lattice spanned by the roots).
- P is the weight lattice of R (containing Q).
- An ordering on the weights: $\mu \le \lambda$ if and only if $\lambda - \mu$ is a nonnegative linear combination of simple roots.
- P^+ is the set of dominant weights: the elements of P in the positive Weyl chamber.
- ρ is the Weyl vector: half the sum of the positive roots; this is a special element of P^+ in the interior of the positive Weyl chamber.
- F is a field of characteristic 0, usually the rational numbers.
- $A = F(P)$ is the group algebra of P, with a basis of elements written e^λ for $\lambda \in P$.
- If $f = e^\lambda$, then $\bar{f}$ means $e^{-\lambda}$, and this is extended by linearity to the whole group algebra.
- $m_\mu = \Sigma_{\lambda \in W\mu} e^\lambda$ is an orbit sum; these elements form a basis for the subalgebra A^W of elements fixed by W.
- $(a;q)_\infty = \prod_{r\ge0}(1-aq^r)$, the infinite q-Pochhammer symbol.
- $\Delta = \prod_{\alpha\in R}\frac{(e^\alpha;q)_\infty}{(te^\alpha;q)_\infty}$.
- $\langle f, g\rangle = (\text{constant term of } f\bar{g}\Delta)/|W|$ is the inner product of two elements of A, at least when t is a positive integer power of q.

The **Macdonald polynomials** P_λ for $\lambda \in P^+$ are uniquely defined by the following two conditions:

$P_\lambda = \sum_{\mu\le\lambda} u_{\lambda\mu} m_\mu$ where $u_{\lambda\mu}$ is a rational function of q and t with $u_{\lambda\lambda} = 1$;

- P_λ and P_μ are orthogonal if $\lambda < \mu$.

Definition 4

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- If we put $t = q\alpha$ and let q tend to 1 the Macdonald polynomials become Jack polynomials (with further conditions)
-

Definition [edit]

The Jack function $J_\kappa^{(\alpha)}(x_1, x_2, \ldots)$ of integer partition κ, parameter α, and indefinitely many arguments $x_1, x_2, \ldots,$ can be recursively defined as follows:

For *m*=1

$$J_k^{(\alpha)}(x_1) = x_1^k(1+\alpha)\cdots(1+(k-1)\alpha)$$

For *m*>1

$$J_\kappa^{(\alpha)}(x_1, x_2, \ldots, x_m) = \sum_\mu J_\mu^{(\alpha)}(x_1, x_2, \ldots, x_{m-1}) x_m^{|\kappa/\mu|} \beta_{\kappa\mu},$$

where the summation is over all partitions μ such that the **skew partition** κ/μ is a **horizontal strip**, namely

$\kappa_1 \geq \mu_1 \geq \kappa_2 \geq \mu_2 \geq \cdots \geq \kappa_{n-1} \geq \mu_{n-1} \geq \kappa_n$ (μ_n must be zero or otherwise $J_\mu(x_1, \ldots, x_{n-1}) = 0$) and

$$\beta_{\kappa\mu} = \frac{\prod_{(i,j)\in\kappa} B_{\kappa\mu}^\kappa(i,j)}{\prod_{(i,j)\in\mu} B_{\kappa\mu}^\mu(i,j)},$$

where $B_{\kappa\mu}^\nu(i,j)$ equals $\kappa_j' - i + \alpha(\kappa_i - j + 1)$ if $\kappa_j' = \mu_j'$ and $\kappa_j' - i + 1 + \alpha(\kappa_i - j)$ otherwise. The expressions κ' and μ' refer to the conjugate partitions of κ and μ, respectively. The notation $(i,j) \in \kappa$ means that the product is taken over all coordinates (i,j) of boxes in the Young diagram of the partition κ.

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- When $\alpha = 1/2$, with some normalization, Jack polynomial gives $\mathcal{C}_p$.

Macdonald polynomial $\longrightarrow$ Jack polynomial $\longrightarrow$ zonal polynomial.

$$ {}_sF_t^{(\alpha)}\begin{pmatrix} a_1,\dots,a_s \\ b_1,\dots,b_t \end{pmatrix} : Y \Bigg) := \sum_{n=0}^{\infty}\sum_{p\in\mathcal{P}_n} \frac{(a_1)_p\cdots(a_s)_p}{(b_1)_p\cdots(b_t)_p}\cdot\frac{\mathcal{C}_p^{(\alpha)}(Y)}{n!} $$