Integer-Valued Polynomials on 3×3 Matrices

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Overview

The Ring of Integer-Valued Polynomials

The set

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

of rational polynomials taking integer values over the integers forms a subring of $\mathbb{Q}[x]$ called the *ring of integer-valued polynomials* (IVPs).

$\mathrm{Int}(\mathbb{Z})$ is a polynomial ring and has basis $\left\{\binom{x}{k} : k \in \mathbb{Z}_{>0}\right\}$ as a $\mathbb{Z}$-module, with

$$\binom{x}{k} := \frac{x(x-1)\cdots(x-(k-1))}{k!}\ , \quad \binom{x}{0} = 1\ , \quad \binom{x}{1} = x\ .$$

This basis is a *regular basis*, meaning that the basis contains exactly one polynomial of degree k for $k \geq 1$.

p-orderings

The study of IVPs on subsets of the integers greatly benefited from the introduction of p-orderings by Bhargava [1].

Definition

Let S be a subset of $\mathbb{Z}$ and p be a fixed prime. A *p-ordering of S* is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ defined as follows: choose an element $a_0 \in S$ arbitrarily. Further elements are defined inductively where, given $a_0, a_1, \ldots, a_{k-1}$, the element $a_k \in S$ is chosen so as to minimize the highest power of p dividing

$$\prod_{i=0}^{k-1} (a_k - a_i) \ .$$

p-sequences

The choice of a p-ordering gives a corresponding sequence:

Definition

The *associated p-sequence of S*, denoted $\{\alpha_{S,p}(k)\}_{k=0}^{\infty}$, is the sequence wherein the k^{th} term $\alpha_{S,p}(k)$ is the power of p minimized at the k^{th} step of the process defining a p-ordering. More explicitly, given a p-ordering $\{a_i\}_{i=0}^{\infty}$ of S,

$$\alpha_{S,p}(k) = \nu_p\left(\prod_{i=0}^{k-1}(a_k - a_i)\right) = \sum_{i=0}^{k-1}\nu_p(a_k - a_i)\ .$$

An Example of p-orderings and p-sequences

Let $p = 2$ and $S = \{1, 2, 3, 5, 8, 13\}$. What is a possible p-ordering for S?

k	0	1	2	3	4	5
a_k	1	2	3	8	5	13
$\alpha_{S,p}(k)$	0	0	1	1	3	6

What happens if we make a different choice for a_0?

k	0	1	2	3	4	5
a_k	5	8	2	3	1	13
$\alpha_{S,p}(k)$	0	0	1	1	3	6

Though the choice of a p-ordering of S is not unique, the associated p-sequence of a subset $S \subseteq \mathbb{Z}$ is independent of the choice of p-ordering [1].

These p-orderings can be used to define a generalization of the binomial polynomials to a specific set $S \subseteq \mathbb{Z}$ which serve as a basis for the integer-valued polynomials of S over Z,

$$\mathrm{Int}(S, \mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(S) \subseteq \mathbb{Z}\} .$$

An analogous definition of P-orderings and P-sequences exists for a subset E of a Dedekind domain D where P is a nonzero prime ideal of D. As for $\mathrm{Int}(S, \mathbb{Z})$, the P-ordering plays a role in determining a regular basis for $\mathrm{Int}(E, D)$, should one exist.

Polynomials over Noncommutative Rings

Let R be any ring, with $R[x]$ the associated polynomial ring, where the variable x commutes elementwise with all of R. Note that though

$$f(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} x^i a_i \ ,$$

the *evaluation* of these two expressions at an element $r \in R$ may be different – that is, it is possible that $\sum_{i=0}^{n} a_i r^i \neq \sum_{i=0}^{n} r^i a_i$.

For this reason, the standard definition of evaluation of a function $f(x)$ at $r \in R$ requires f to be expressed in the form $\sum_{i=0}^{n} a_i x^i$, and then substituting r for x.

Polynomials over Division Rings

Theorem (Gordon-Motzkin, [5] 16.4)

Let D be a division ring, and let f be a polynomial of degree n in $D[x]$. Then the roots of f lie in at most n conjugacy classes of D. This means that if $f(x) = (x - a_1) \cdots (x - a_n)$ with $a_1, \ldots, a_n \in D$, then any root of f is conjugate to some a_i.

Theorem (Dickson's Theorem, [5] 16.8)

Let D be a division ring and F its centre. Let $a, b \in D$ be two elements that are algebraic over F. Then a and b are conjugate in D if and only if they have the same minimal polynomial over F.

A theorem of Bray-Whaples ([5], 16.13) purports that there is such thing as a minimal polynomial over a set of elements in a division ring. The construction for such a polynomial is given by the following proposition.

Proposition ([4], 2.4)

Let D be a subring of a division algebra, and $c_1, \ldots, c_n$ be n pairwise nonconjugate elements of D. Then the minimal polynomial is given inductively by

$$\begin{aligned} f(a_0)(x) &= (x - a_0) \\ f(a_0, \ldots, a_n)(x) &= (x - a_n^{f(a_0,\ldots,a_{n-1})(a_n)}) \cdot f(a_0, \ldots, a_{n-1})(x) \ . \end{aligned}$$

Maximal Orders

Definition ([6], Section 8)

Let R be a Noetherian integral domain with quotient field K, and A a finite-dimensional K-algebra. An *R-order* in A is a subring Λ of A which has the same unit element as A, and is such that Λ is a finitely-generated R-submodule with $K \cdot \Lambda = A$.

Note that every finite-dimensional K-algebra A contains R-orders, since there exist $y_1, y_2, \ldots, y_n \in A$ such that $A = \sum_{i=1}^{n} K y_i$, and so $\Delta = \sum_{i=1}^{n} R y_i$ will satisfy the definition of an R-order.

Definition ([6])

A *maximal R-order* in A is an R-order which is not properly contained in any other R-order in A.

Constructing a Maximal Order

When R is a complete DVR with unique maximal ideal P, R/P is finite, K is the quotient field of R, D is a division ring with centre containing K, and $[D : K] = n^2$, then D contains a unique maximal R-order Δ and we can explicitly describe the structures of the division ring D and maximal order Δ, via a construction given in Reiner [6]. Furthermore, the description of the structure can be chosen to only depend on n.

For the sake of simplicity and future reference, here we describe the construction only in the case that $|R/P| = 2$ and $n = 3$, and in minimal detail.

Let ω be a primitive 7^{th} root of unity, and let $W = \mathbb{Q}_2(\omega)$. Define elements

$$\omega^* = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix} \qquad \pi_D^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} .$$

Then the map generated by $\omega \mapsto \omega^*$ defines a $\mathbb{Q}_2$-isomorphism $W \to W^* = \mathbb{Q}_2(\omega^*) \subseteq M_3(\mathbb{Q}_2(\omega))$, under which scalars $\lambda \in \mathbb{Q}_2$ are identified with $\lambda I_3 \in M_3(\mathbb{Q}_2)$.

The following relations exist between ω^* and π_D^*:

$$(\pi_D^*)^3 = 2I_3 \qquad\qquad \pi_D^* \cdot \omega^* = (\omega^*)^2 \cdot \pi_D^*$$

We then define

$$D = \mathbb{Q}_2[\omega^*, \pi_D^*] \ ,$$

which is a division ring with centre containing $\mathbb{Q}_2$ and $[D : \mathbb{Q}_2] = 9 = 3^2$. The maximal order in D is

$$\Delta = \mathbb{Z}_2[\omega^*, \pi_D^*] \ .$$

IVPs over Matrix Rings

We are particularly interested in studying IVPs over matrix rings.

We denote the set of rational polynomials mapping integer matrices to integer matrices by

$$\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z})) = \{f \in \mathbb{Q}[x] : f(M) \in M_n(\mathbb{Z}) \text{ for all } M \in M_n(\mathbb{Z})\}\ .$$

We know from Cahen and Chabert [2] that $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ has a regular basis, but it is not easy to describe using a formula in closed form [3].

Finding a regular basis for $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ is related to finding a regular basis for its integral closure. In order to study the latter object, we would like to describe the localizations of the integral closure of $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ at rational primes. To do this, we can use results about division algebras over local fields.

Theorem (in appendix of [7])

If D is a division algebra of degree n^2 over a local field K and F is a field extension of degree n of K, then F can be embedded as a maximal commutative subfield of D.

If p is a fixed prime, D is a division algebra of degree n^2 over $K = \mathbb{Q}_p$, and R_n is its maximal order, then we obtain the following useful result:

Proposition ([3], 2.1)

The integral closure of $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z})_{(p)})$ is $\text{Int}_{\mathbb{Q}}(R_n)$.

Thus, the problem of describing the integral closure of $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z})_{(p)})$ is exactly that of describing $\text{Int}_{\mathbb{Q}}(R_n)$, and so we move our attention towards studying IVPs over maximal orders.

An Analogue to p-orderings

Definition ([4], 1.1)

Let K be a local field with valuation ν, D be a division algebra over K to which ν extends, R the maximal order in D, and S a subset of R. Then a *ν-ordering* of S is a sequence $\{a_i : i = 0, 1, 2, \dots\} \subseteq S$ such that for each $k > 0$, the element a_k minimizes the quantity $\nu(f_k(a_0, \dots, a_{k-1})(a))$ over $a \in S$, where $f_k(a_0, \dots, a_{k-1})(x)$ is the minimal polynomial of the set $\{a_0, a_1, \dots, a_{k-1}\}$, with the convention that $f_0 = 1$. We call the sequence of valuations $\{\nu(f_k(a_0, \dots, a_{k-1})(a_k)) : k = 0, 1, \dots\}$ the *ν-sequence* of S.

Proposition ([4], 1.2)

Let K be a local field with valuation ν, D be a division algebra over K to which ν extends, R the maximal order in D, and S a subset of R. Additionally, let $\pi \in R$ be a uniformizing element, meaning an element for which $(\pi^n) = (p)$, let $\{a_i : i = 0, 1, 2, \dots\} \subseteq S$ be a ν-ordering, and let $f_k(a_0, \dots, a_{k-1})$ be the minimal polynomial of $\{a_0, a_1, \dots, a_{k-1}\}$. Then the sequence $\{\alpha_S(k) = \nu(f_k(a_0, \dots, a_{k-1})(a_k)) : k = 0, 1, 2, \dots\}$ depends only on the set S, and not on the choice of ν-ordering. The sequence of polynomials

$$\{\pi^{-\alpha_S(k)} f_k(a_0, \dots, a_{k-1})(x) : k = 0, 1, 2, \dots\}$$

forms a regular R-basis for the R-algebra of polynomials which are integer-valued on S.

In order to use this proposition, we need to be able to construct a ν-ordering for the maximal order R_n. A recursive method for constructing ν-orderings for elements of a maximal order is based on two lemmas.

Lemma (see [4], 6.2)

Let $\{a_i : i = 0, 1, 2, \dots\}$ be a ν-ordering of a subset S of R with associated ν-sequence $\{\alpha_S(i) : i = 0, 1, 2, \dots\}$ and let b be an element in the centre of R. Then:

i) $\{a_i + b : i = 0, 1, 2, \dots\}$ is a ν-ordering of $S + b$, and the ν-sequence of $S + b$ is the same as that of S

ii) If p is the characteristic of the residue field of K (so that $(p) = (\pi)^n$ in R), then $\{pa_i : i = 0, 1, 2, \dots\}$ is a ν-ordering for pS and the ν-sequence of pS is $\{\alpha_S(i) + in : i = 0, 1, 2, \dots\}$

Definition

The *shuffle* of two nondecreasing sequences of integers is their disjoint union sorted into nondecreasing order. If the sequences are $\{b_i\}$ and $\{c_i\}$, their shuffle is denoted $\{b_i\} \wedge \{c_i\}$.

Lemma ([4], 5.1)

Let R be a commutative ring with S a subset of R. Let S_1 and S_2 be disjoint subsets of S with the property that $\nu(s_1 - s_2) = 0$ for any $s_1 \in S_1$ and $s_2 \in S_2$, and that S_1 and S_2 are each closed with respect to conjugation by elements of R. If $\{b_i\}$ and $\{c_i\}$ are ν-orderings of S_1 and S_2 respectively with associated ν-sequence $\{\alpha_{S_1}(i)\}$ and $\{\alpha_{S_2}(i)\}$, then the ν-sequence of $S_1 \cup S_2$ is the shuffle $\{\alpha_{S_1}(i)\} \wedge \{\alpha_{S_2}(i)\}$, and this shuffle applied to $\{b_i\}$ and $\{c_i\}$ gives a ν-ordering of $S_1 \cup S_2$.

Lemma ([4], 5.2)

Let S_1 and S_2 be disjoint subsets of S with the property that there is a non-negative integer k such that $\nu(s_1 - s_2) = k$ for any $s_1 \in S_1$ and $s_2 \in S_2$, and that S_1 and S_2 are each closed with respect to conjugation by elements of R. If $\{b_i\}$ and $\{c_i\}$ are ν-orderings of S_1 and S_2 respectively with associated ν-sequence $\{\alpha_{S_1}(i)\}$ and $\{\alpha_{S_2}(i)\}$, then the ν-sequence of $S_1 \cup S_2$ is the sum of the linear sequence $\{ki : i = 0, 1, 2, \dots\}$ with the shuffle $\{\alpha_{S_1}(i) - ki\} \wedge \{\alpha_{S_2}(i) - ki\}$, and this shuffle applied to $\{b_i\}$ and $\{c_i\}$ gives a ν-ordering of $S_1 \cup S_2$.

The theory presented in the previous slides is utilized by Evrard and Johnson [3] to construct a ν-order for R_2 and establish a ν-sequence and regular basis for the IVPs on R_2 when the division algebra D is over the local field $\mathbb{Q}_2$.

We would like to extend these results to find a regular basis for IVPs on R_3 over the local field $\mathbb{Q}_2$, and further on to all R_n over $\mathbb{Q}_2$.

The Maximal Order

As introduced in previous slides, we are working within the division algebra D and its maximal order Δ, defined as subsets of the 3×3 complex matrices as

$$D = \mathbb{Q}_2[\omega^*, \pi_D^*] \qquad \Delta = \mathbb{Z}_2[\omega^*, \pi_D^*]$$

where $\mathbb{Q}_2, \mathbb{Z}_2$ denote the 2-adic numbers and integers, respectively, and

$$\omega^* = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix} \qquad \pi_D^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

with $\omega = \zeta_7$ a primitive 7^{th} root of unity.

We will abuse notation and use ω to refer to the 3×3 matrix ω^*, and use π to denote π_D^*. Note that we have the relations $\pi^3 = 2I_3$ and $\pi \cdot \omega \cdot \pi^{-1} = \omega^2$, and also that we work with the conventions that, where ω is regarded as a root of unity,

$$\omega + \omega^2 + \omega^4 \equiv 0 \pmod 2 \quad \text{and} \quad \omega^3 + \omega^5 + \omega^6 \equiv 1 \pmod 2 .$$

We also have a valuation ν in Δ described by $\nu(z) = \nu_2(\det(z))$ for $z \in \Delta$ realized as a matrix, when ν_2 denotes the 2-adic valuation.

Conjugacy Classes mod π

Looking at all elements of $\Delta = \mathbb{Z}_2[\omega, \pi_D]$ modulo π, we obtain four conjugacy classes:

$$\begin{aligned}
T &= \{z \in \Delta : z \equiv 0 \pmod{\pi}\} \\
T+1 &= \{z \in \Delta : z \equiv I_3 \pmod{\pi}\} \\
S &= \{z \in \Delta : z \equiv \omega \text{ or } \omega^2 \text{ or } \omega^4 \pmod{\pi}\} \\
S+1 &= \{z \in \Delta : z \equiv \omega^3 \text{ or } \omega^6 \text{ or } \omega^5 \pmod{\pi}\} \\
&= \{z \in \Delta : z \equiv \omega + I_3 \text{ or } \omega^2 + I_3 \text{ or } \omega^4 + I_3 \pmod{\pi}\}
\end{aligned}$$

Since $T+1$ and $S+1$ are translates of T and S, respectively, a previous lemma states that they have the same ν-sequence, so we only need to determine α_T and α_S in order to find a formula for α_Δ.

Conjugacy Classes mod π^2

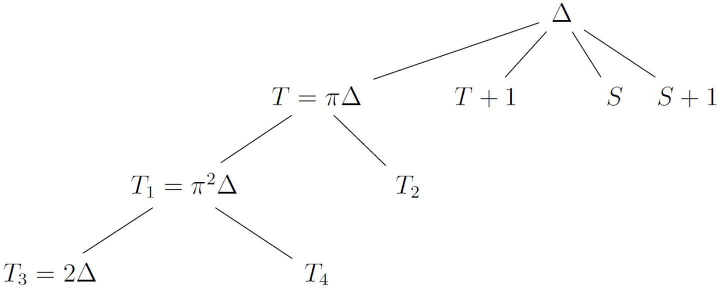
We can break the set T down further by considering conjugacy classes modulo π^2:

$$T_1 = \{z \in \Delta : z \equiv 0 \pmod{\pi^2}\} = \pi^2\Delta$$
$$T_2 = \{z \in \Delta : z \equiv \omega^i\pi \pmod{\pi^2} \text{ for some } 0 \leq i \leq 6\}$$

The set T_1 can be broken down further still by looking at conjugacy classes modulo $\pi^3 = 2$:

$$T_3 = \{z \in \Delta : z \equiv 0 \pmod{\pi^3}\} = 2\Delta$$
$$T_4 = \{z \in \Delta : z \equiv \omega^i\pi^2 \pmod{\pi^3} \text{ for some } 0 \leq i \leq 6\}$$

From this analysis, we obtain the following tree of subsets of Δ:



These sets all satisfy the necessary lemmas pertaining to shuffles of ν-sequences, and so we can derive a formula for α_Δ that depends only on itself, α_S, α_{T_2}, and α_{T_4}.

The ν-sequence of Δ

Based on the tree of subsets and the lemmas, we obtain the following result.

Proposition

The ν-sequence of Δ, denoted α_Δ, satisfies and is determined by the formula

$$\alpha_\Delta = \left(\left[\left(\left[\left(\alpha_\Delta + (n)\right) \wedge \left(\alpha_{T_4} - (2n)\right)\right] + (n)\right) \wedge \left(\alpha_{T_2} - (n)\right)\right] + (n)\right)^{\wedge 2} \wedge \left(\alpha_S\right)^{\wedge 2},$$

where (kn) denotes the linear sequence whose n^{th} term is kn.

It remains to determine the ν-sequences for S, T_2, and T_4.

Characteristic Polynomials

To do so, it is useful to describe the sets S, T_2, and T_4 in terms of their characteristic polynomials.

Given a complex matrix $A \in M_3(\mathbb{C})$, we define the characteristic polynomial of A to be

$$x^3 - Tr(A)x^2 + \beta(A)x - \det(A)$$

where $Tr(A)$ and $\det(A)$ are the usual trace and determinant of a 3×3 matrix, and $\beta(A)$ is defined in terms of the 2×2 minors of A.

Lemma

$$S = \{z \in \Delta : Tr(z) \equiv 0 \ (mod\ 2),\ \beta(z) \equiv 1 \ (mod\ 2), \det(z) \equiv 1 \ (mod\ 2)\}$$
$$T_2 = \{z \in \Delta : Tr(z) \equiv 0 \ (mod\ 2),\ \beta(z) \equiv 0 \ (mod\ 2), \det(z) \equiv 2 \ (mod\ 4)\}$$
$$T_4 = \{z \in \Delta : Tr(z) \equiv 0 \ (mod\ 2),\ \beta(z) \equiv 0 \ (mod\ 4), \det(z) \equiv 4 \ (mod\ 8)\}$$

We can determine some useful facts about the valuation of certain polynomials within S, T_2, and T_4, with the goal of establishing these as the minimal polynomials within their respective sets. This process is analogous to the one presented in Evrard and Johnson [3] and Johnson [4].

A Polynomial in S

Let us define the function

$$\phi = (\phi_1, \phi_2, \phi_3) : \mathbb{Z}_{\geq 0} \to 2\mathbb{Z}_{\geq 0} \times (1 + 2\mathbb{Z}_{\geq 0}) \times (1 + 2\mathbb{Z}_{\geq 0})$$

$$\phi(n) = \left(2\sum_{i\geq 0} n_{3i}2^i, 1 + 2\sum_{i\geq 0} n_{3i+1}2^i, 1 + 2\sum_{i\geq 0} n_{3i+2}2^i \right)$$

where $n = \sum_{i\geq 0} n_i 2^i$ is the expansion of n in base 2. Let

$$f_n(x) = \prod_{k=0}^{n-1} \left(x^3 - \phi_1(k)x^2 + \phi_2(k)x - \phi_3(k)\right) .$$

Lemma

If $z \in S$ then

$$\nu(f_n(z)) \geq 3n + 3\sum_{i>0} \left\lfloor \frac{n}{8^i} \right\rfloor$$

with equality if $Tr(z) = \phi_1(n)$, $\beta(z) = \phi_2(n)$, and $\det(z) = \phi_3(n)$.

A Polynomial in T_4

Let us define the function

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) : \mathbb{Z}_{\geq 0} \to 2\mathbb{Z}_{\geq 0} \times 4\mathbb{Z}_{\geq 0} \times (4 + 8\mathbb{Z}_{\geq 0})$$

$$\sigma(n) = \left(2\sum_{i\geq 0} n_{3i}2^i, 4\sum_{i\geq 0} n_{3i+1}2^i, 4 + 8\sum_{i\geq 0} n_{3i+2}2^i \right)$$

where $n = \sum_{i\geq 0} n_i 2^i$ is the expansion of n in base 2. Let

$$h_n(x) = \prod_{k=0}^{n-1} \left(x^3 - \sigma_1(k)x^2 + \sigma_2(k)x - \sigma_3(k) \right) .$$

Lemma

If $z \in T_4$ then

$$\nu(h_n(z)) \geq 7n + \sum_{i>0} \left\lfloor \frac{n}{2^i} \right\rfloor$$

with equality if $Tr(z) = \sigma_1(n)$, $\beta(z) = \sigma_2(n)$, and $\det(z) = \sigma_3(n)$.

A Polynomial in T_2

Let us define the function

$$\psi = (\psi_1, \psi_2, \psi_3) : \mathbb{Z}_{\geq 0} \to 2\mathbb{Z}_{\geq 0} \times 2\mathbb{Z}_{\geq 0} \times (2 + 4\mathbb{Z}_{\geq 0})$$

$$\psi(n) = \left(2\sum_{i\geq 0} n_{3i+1}2^i, 2\sum_{i\geq 0} n_{3i}2^i, 2 + 4\sum_{i\geq 0} n_{3i+2}2^i\right)$$

where $n = \sum_{i\geq 0} n_i 2^i$ is the expansion of n in base 2. Let

$$g_n(x) = \prod_{k=0}^{n-1} \left(x^3 - \psi_1(k)x^2 + \psi_2(k)x - \psi_3(k)\right) .$$

Lemma

If $z \in T_2$ then

$$\nu(g_n(z)) \geq 4n + \sum_{i>0} \left\lfloor \frac{n}{2^i} \right\rfloor$$

with equality if $Tr(z) = \psi_1(n)$, $\beta(z) = \psi_2(n)$, and $\det(z) = \psi_3(n)$.

Extension to General n

This construction can of course be extended to any subset S of a maximal order Δ sitting in $M_n(\mathbb{Q}_2)$ that is closed under conjugation, but the practical use of the construction comes from the fact that it is possible to achieve a known minimum when taking the valuation of the polynomials generated.

For any valuation ν, if the valuation of n terms $a_1, \ldots, a_n$ produces a complete set of residues modulo n, then it must be the case that $\nu(a_1 + \cdots + a_n) = \min_{1 \le i \le n} \nu(a_i)$. This fact is applied in the valuation of the polynomial

$$f(z) = z^n - \phi_1(k)z^{n-1} + \phi_2(k)z^{n-2} + \cdots + (-1)^n \phi_n(k)$$

with $z \in S$ to show that a minimum for $\nu(f)$ can be determined with certainty only when $\gcd(n, \nu(z)) = 1$.

In particular, this means that this construction should work for the case of the $q \times q$ matrices, where $q = n$ is prime. It should also work for some subsets of Δ when n is composite. It remains to see what adjustments must be made to this construction in the case where n is composite, and if there is any difference between the case where n is a power of a prime or n is squarefree.

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