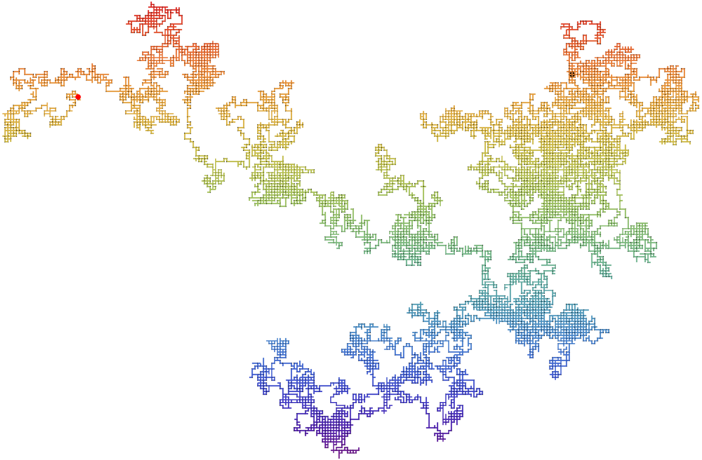
Hidden Walks

Lin Jiu

Dalhousie University Number Theory Seminar

Feb. 26, 2018

Introduction



Outline

Motzkin Path

Generalized Euler Polynomial

Harmonic Sums

Joint Work with

Diane Shi

Generalized Motzkin Numbers

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_k M_{n,k+1}$$

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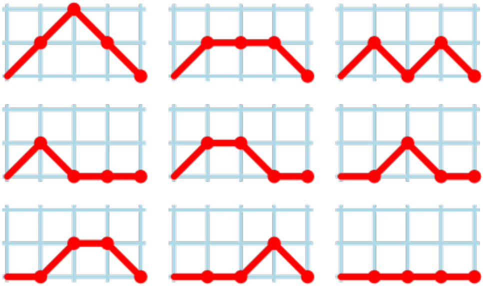

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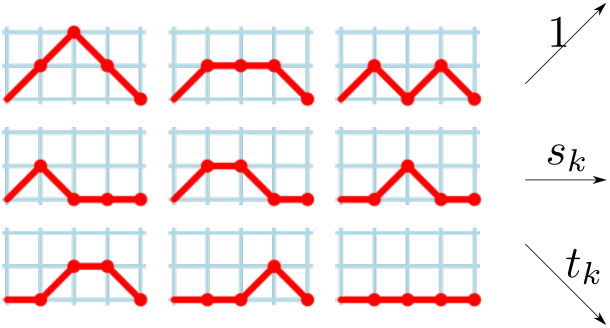

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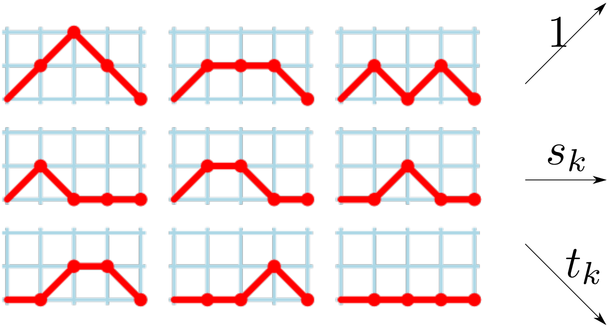

$$\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}$$

Bernoulli and Euler Polynomials

Bernoulli $(B_n(x))_{n=0}^{\infty}$ and Euler polynomials $(E_n(x))_{n=0}^{\infty}$:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \text{ and } \frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

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Theorem(L. Jiu and D. Shi)

Let $w_k^{(1)} = -\frac{k^4}{4(2k+1)(2k-1)}$ and $w_k^{(2)} = -\frac{k^2}{4}$. Define $(M_{n,k}(x))_{n,k=0}^{\infty}$ by the recurrence

$$M_{n+1,k}(x) = M_{n,k-1}(x) + \left(x - \frac{1}{2}\right) M_{n,k}(x) + w_k^{(j)} M_{n,k+1}(x),$$

together with initials $M_{0,0}(x) = 1$ and $M_{n,k}(x) = 0$ if $k > n$. Then, when $k = 0$, we have

$$M_{n,0}(x) = \begin{cases} B_n(x), & \text{if } j = 1; \\ E_n(x), & \text{if } j = 2. \end{cases}$$

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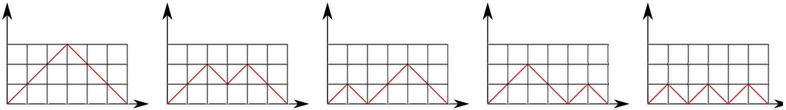
Dyck path:



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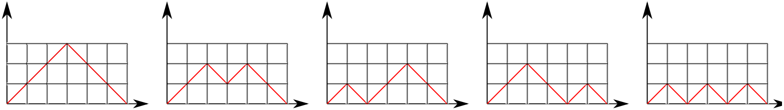

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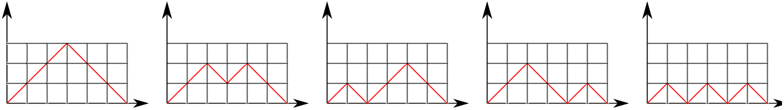

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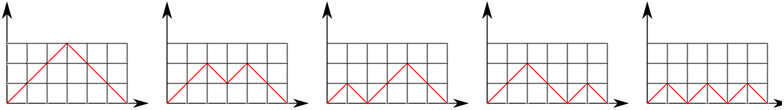

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Recall

$$(\nearrow, \rightarrow, \searrow) = (1, s_k, t_k) = \left(1, 0, -\frac{k^2}{4}\right)$$

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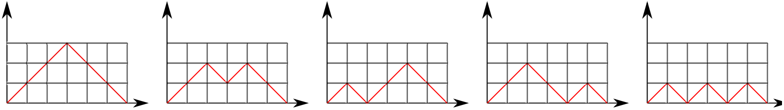

$$-\frac{61}{64} = \frac{E_6}{2^6} = \left(\frac{-3^2}{4}\right)\left(\frac{-2^2}{4}\right)\left(\frac{-1^2}{4}\right) + \left(\frac{-2^2}{4}\right)^2\left(\frac{-1^2}{4}\right) + \left(\frac{-1^2}{4}\right)\left(\frac{-2^2}{4}\right)\left(\frac{-1^2}{4}\right)$$
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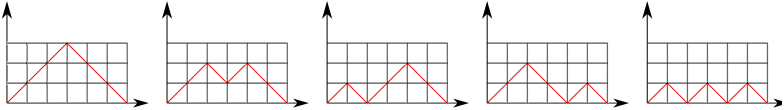

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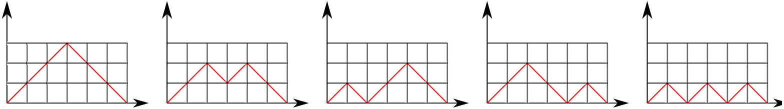

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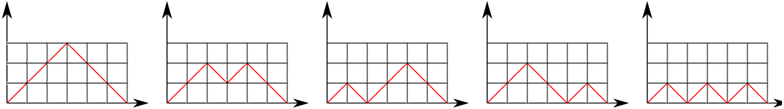

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- $E_{2n+1} = 0$ and $E_{2n} \in \mathbb{Z}$;
- $(-1)^n E_{2n} > 0$.

n	0	1	2	3	4	5	6	7	8
E_n	1	0	-1	0	5	0	-61	0	1385

Joint Work with

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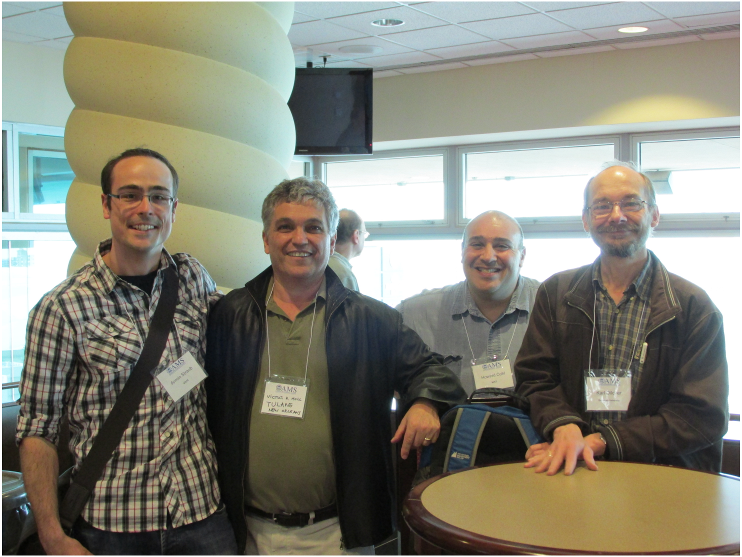
C. Vignat

V. H. Moll


AMS
Armin Straub
UIUC
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Question: inverse formula

$$E_n(x) = f\left(E_{n_1}^{(p_1)}(x),\ldots,E_{n_k}^{(p_k)}(x)\right)?$$

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Theorem(L. Jiu, V. H. Moll and C. Vignat)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

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$N = 2$:

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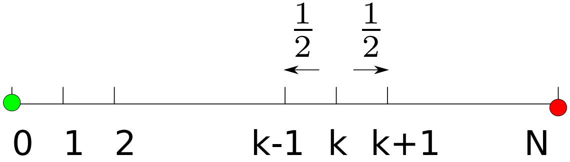

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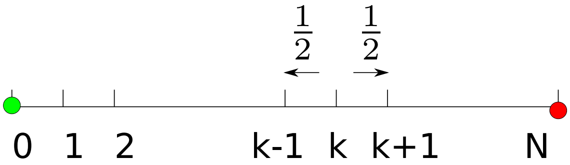

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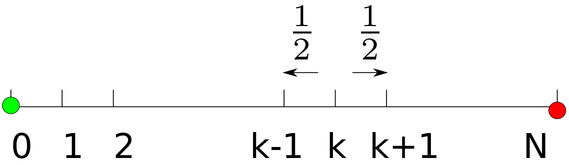

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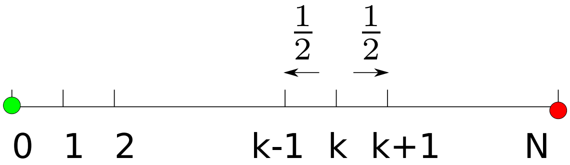

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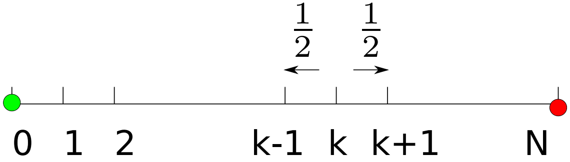

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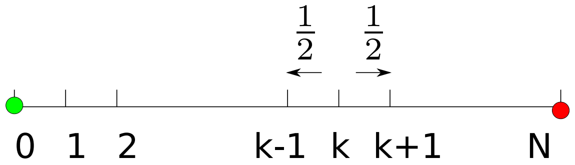

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$$p_l^{(N)} = P(\nu_N = l) = \frac{1}{N} \sum_{k=1}^{N} (-1)^{k+1} \sin\left(\theta_k^{(N)}\right) \cos^{\ell-1}\left(\theta_k^{(N)}\right),$$

where

$$\theta_k^{(N)} := \frac{\pi(2k-1)}{2N}.$$

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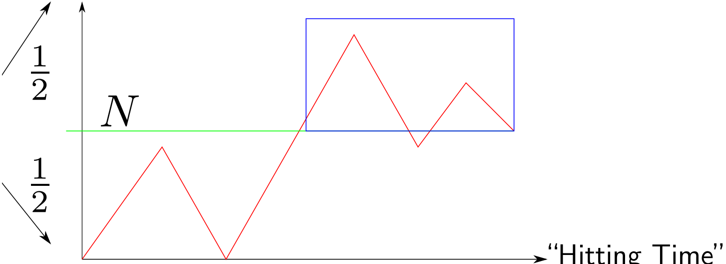
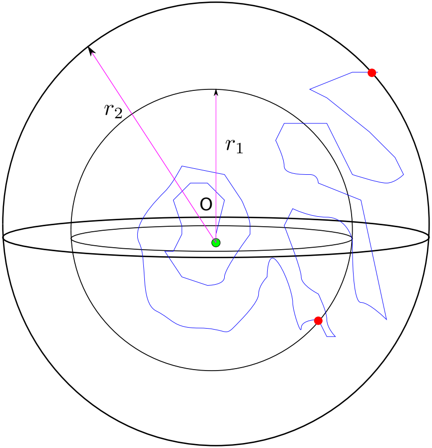
Remark



3-dim



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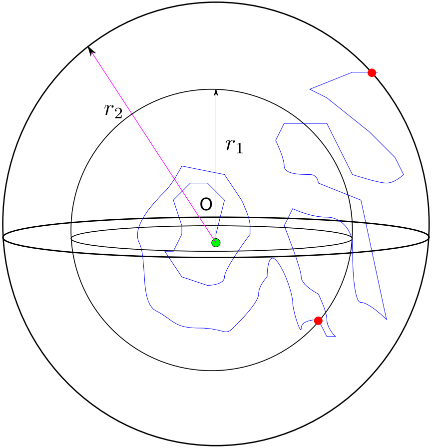

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- Example $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$,

$$\frac{3^n}{n+1}\left[B_{n+1}\left(\frac{x+5}{6}\right) - B_{n+1}\left(\frac{x+3}{6}\right)\right]$$
$$= \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4^k}E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

Harmonic sums

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$$\mathbb{P}(6 \to 6) = \cdots = \mathbb{P}(6 \to 1) = \frac{1}{6}.$$

Walk

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STEP 1: walk $N \to n_1 \, (\leq N)$ with $\mathbb{P}\,(N \to n_1) = \frac{1}{N}$;

Walk

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N-1 & N \end{array}$$

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STEP 2: walk $n_1 \to n_2 \, (\leq n_1)$, with $\mathbb{P}\,(n_1 \to n_2) = \frac{1}{n_1}$;

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<u>STEP $k+1$</u>: walk $n_k \mapsto n_{k+1}\,(\leq n_k)$, with $\mathbb{P}\,(n_k \to n_{k+1}) = \frac{1}{n_k}$.

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... ...

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$$\mathbb{P}\,(n_{k+1} = 1) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k} = \frac{S_{\underbrace{1,\ldots,1}_{k}}(N)}{N}.$$

Transition matrix

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$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

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$$M_{n+1,k}(x) = M_{n,k-1}(x) + \left(x - \frac{1}{2}\right) M_{n,k}(x) - \frac{k^2}{4} M_{n,k+1}(x),$$

where $M_{n,0}(x) = E_n(x)$.

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$$R = \begin{pmatrix} x-\frac{1}{2} & -\frac{1}{4} & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{1}{2} & -1 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\frac{k^2}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{1}{2} & -\frac{(k+1)^2}{4} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix}$$

End