Irreducibility and Roots of a Class of Polynomials

(Number Theory Seminar)

Abdullah Al-Shaghay

Dalhousie University

Monday March 5, 2018

Overview

Overview

1. Introduction
2. What About a Gap?
3. Pictures
4. Future Directions

Harrington

In a 2012 paper, J. Harrington investigated the factorization properties of polynomials of the form

$$f(x) = x^n + cx^{n-1} + cx^{n-2} + \ldots + cx + c \in \mathbb{Z}[x].$$

In particular, it was asked:

- For what positive integers n and nonzero integers c is $f(x)$ irreducible?

Harrington

In a 2012 paper, J. Harrington investigated the factorization properties of polynomials of the form

$$f(x) = x^n + cx^{n-1} + cx^{n-2} + \ldots + cx + c \in \mathbb{Z}[x].$$

In particular, it was asked:

- For what positive integers n and nonzero integers c is $f(x)$ irreducible?
- If $f(x)$ is reducible, then how does it factor?

Special Cases

Certain cases of this problem have already been answered:

- If there exists a prime p such that $p || c$, then $f(x)$ is irreducible for all values n by the Eisenstein criterion.

Special Cases

Certain cases of this problem have already been answered:

- If there exists a prime p such that $p||c$, then $f(x)$ is irreducible for all values n by the Eisenstein criterion.
- If $c = 1$, $f(x)$ is irreducible if and only if $n = p - 1$ for an odd prime p.

Main Result

Theorem

Let n and c be positive integers with $c \geq 2$. Then the polynomials

$$f(x) = x^n + \sum_{j=0}^{n-1} cx^j, \qquad g(x) = x^n + \sum_{j=0}^{n-1} (-1)^{n-j} cx^j,$$

$$h(x) = x^n - \sum_{j=0}^{n-1} cx^j, \qquad k(x) = x^n - \sum_{j=0}^{n-1} (-1)^{n-j} cx^j,$$

are irreducible in $\mathbb{Z}[x]$ with the exceptions of:
$f(x) = x^2 + 4x + 4 = (x+2)^2$ *and* $g(x) = x^2 - 4x + 4 = (x-2)^2$.

"Behind the Scenes"

Theorem

Let n, c, and d be positive integers with $n \geq 3, d \neq c, d \leq 2(c-1)$, and $(n, c) \neq (3, 3)$. If the trinomial $f(x) = x^n \pm cx^{n-1} \pm d$ is reducible in $\mathbb{Z}[x]$, then $f(x) = (x \pm 1)g(x)$ for some irreducible $g(x) \in \mathbb{Z}[x]$.

Theorem (Rouché's Theorem)

Given two functions f and g analytic inside some region K with a closed contour as ∂K, if $|g(z)| < |f(z)|$ on ∂K, then f and $f + g$ have the same number of zeros inside K (counting multiplicity).

Lemma

Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with $f(0) \neq 0$. If $f(x)$ has only one root (counting multiplicity) in $\mathbb{C} \backslash \mathcal{U}$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Lemma

Let $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 \in \mathbb{Q}[x]$ with $n \geq 1$ and $a_0 \neq 0$. If $|a_{n-1}| > 1 + \sum_{j=0}^{n-2} |a_j|$, then $f(x)$ has exactly one root (counting multiplicity) in $\mathbb{C} \backslash \mathcal{U}$.

Lemma

Let n, m, c, and d be positive integers with $n > m$ and $d \geq c + 1$. Then the trinomial $f(x) = x^n \pm cx^m \pm d$ has no roots in $\mathcal{U}$. Furthermore, if $d > c + 1$, then $f(x)$ has no roots in $\overline{\mathcal{U}}$.

Lemma

Let n, c and d be positive integers with $n \geq 2$ and $d < (c-1)^{n-1}$. Then the trinomial $f(x) = x^n \pm cx^{n-1} \pm d$ has a root $\alpha \in \mathbb{R}$ with $|\alpha| > c - 1$.

Lemma

Let K be a positive integer and let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with no roots in the set $\{z \in \mathbb{C} : |z| \leq K\}$. If $f(x)$ has a root α with $|\alpha| > \frac{|f(0)|}{K+1}$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Cases are then cosidered:

- $d < c - 1$ and use first 2 lemmas.

Lemma

Let K be a positive integer and let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with no roots in the set $\{z \in \mathbb{C} : |z| \leq K\}$. If $f(x)$ has a root α with $|\alpha| > \frac{|f(0)|}{K+1}$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Cases are then cosidered:

- $d < c - 1$ and use first 2 lemmas.
- $c + 1 < d \leq 2(c - 1)$ and use the last 3 lemmas.

Lemma

Let K be a positive integer and let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with no roots in the set $\{z \in \mathbb{C} : |z| \leq K\}$. If $f(x)$ has a root α with $|\alpha| > \frac{|f(0)|}{K+1}$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.

Cases are then cosidered:

- $d < c - 1$ and use first 2 lemmas.
- $c + 1 < d \leq 2(c - 1)$ and use the last 3 lemmas.
- $d = c \pm 1$ are dealt with separately.

Overview

New Polynomials, Old Questions

$$f(x) = x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c \in \mathbb{Z}[x].$$

- For what positive integers n and c is $f(x)$ irreducible?

New Polynomials, Old Questions

$$f(x) = x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c \in \mathbb{Z}[x].$$

- For what positive integers n and c is $f(x)$ irreducible?
- If $f(x)$ is reducible, then how does it factor?

Analogous Conjecture

Conjecture

Let n, c, and a be positive integers. If the quadranomial $h(x) = x^{n+1} - x^n + cx^{n-a} - c$ is reducible in $\mathbb{Z}[x]$, then $h(x) = (x-1)g(x)$ for some irreducible $g(x) \in \mathbb{Z}[x]$.

Irreducibility

Theorem

Let n, a, and c be positive integers with $c \geq 2$. Then the polynomials $f(x) = x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c$ are irreducible in $\mathbb{Z}[x]$.

Overview

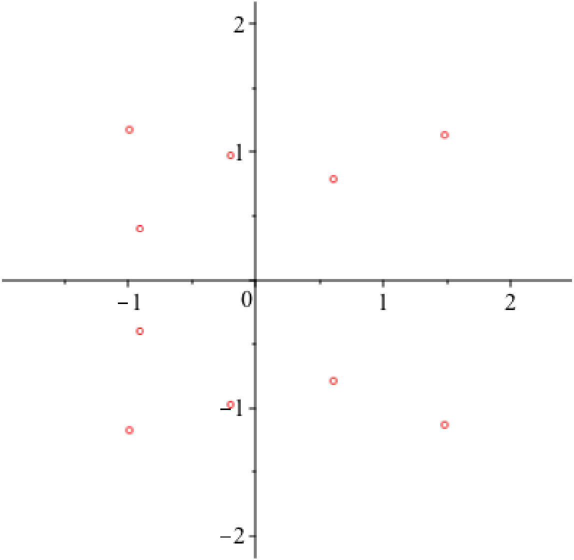
Root Behaviour

Let us take a look at the roots of

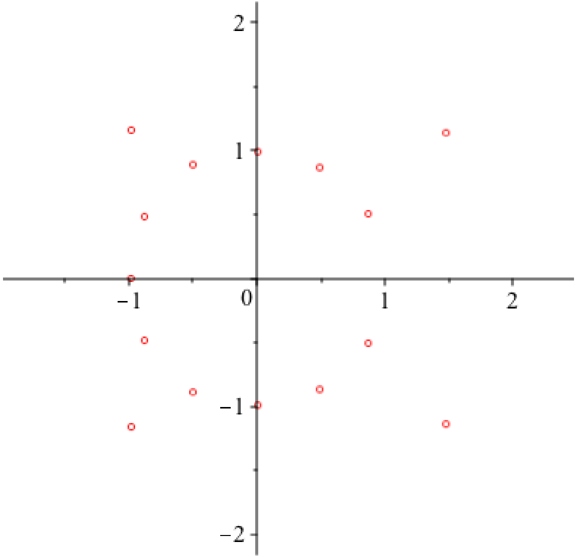
$$x^n + cx^{n-a-1} + cx^{n-a-2} + \ldots + cx + c = f_{n,a,c}(x)$$

for different values of n, a, and c.

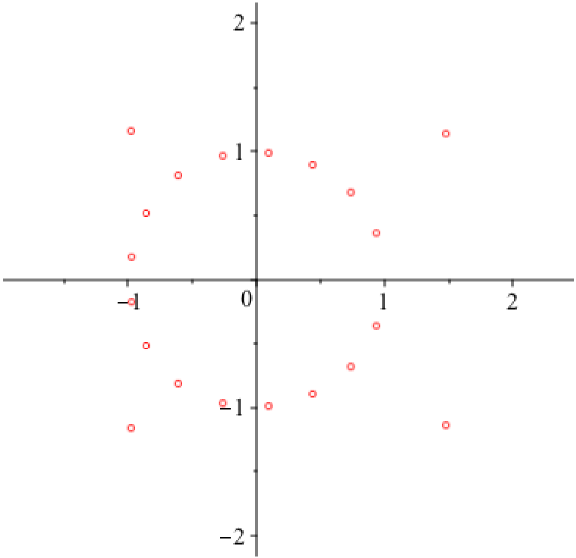
$(n, a, c) = (10, 3, 8)$



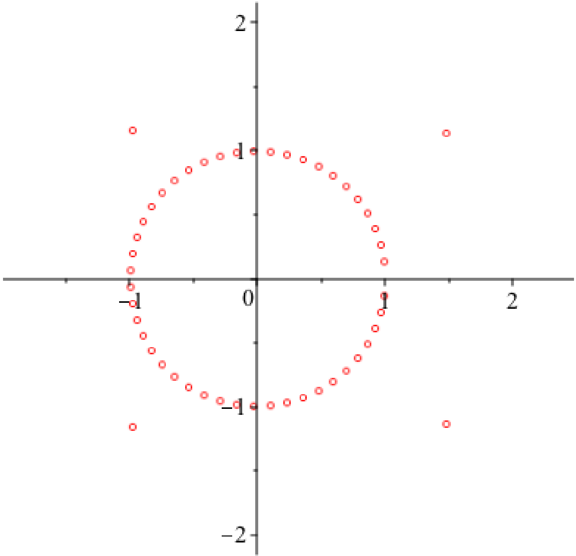
$(n, a, c) = (15, 3, 8)$



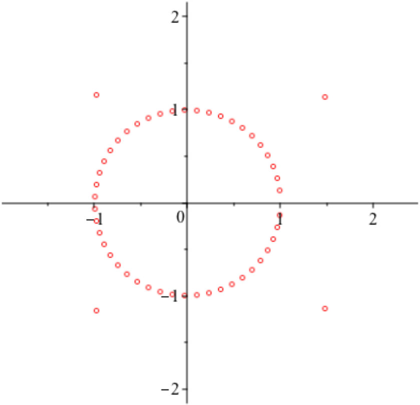
$(n, a, c) = (20, 3, 8)$

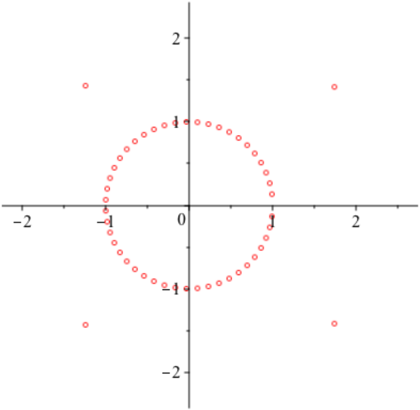


$(n, a, c) = (50, 3, 8)$

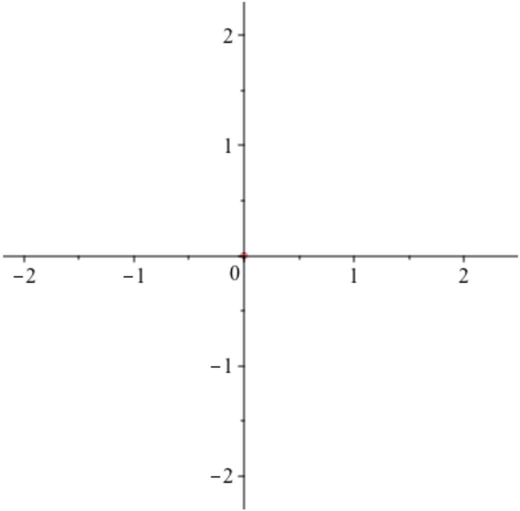


$(n, a, c) = (50, 3, 8)$ vs. $(n, a, c) = (50, 3, 18)$

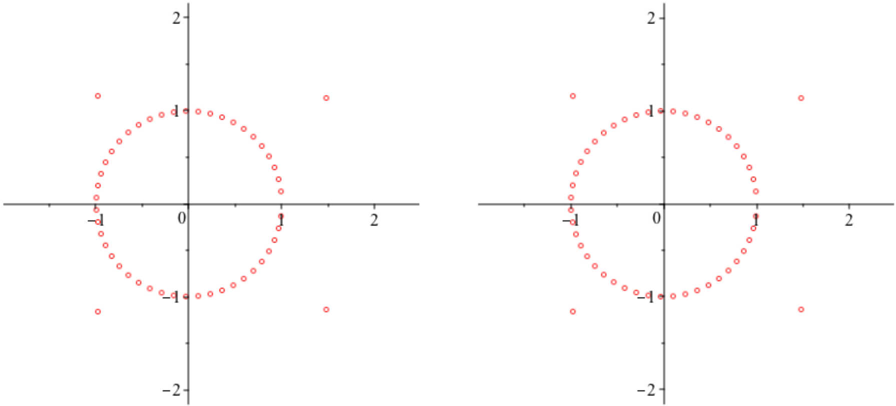




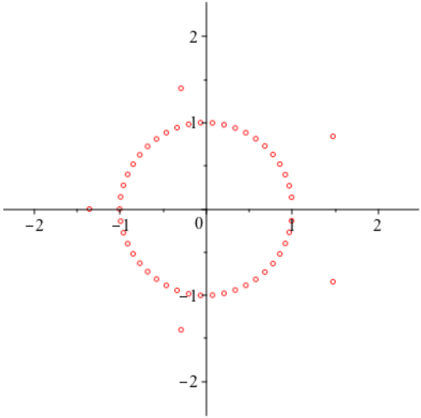
$(n, 3, 8) \quad 1 \leq n \leq 50$

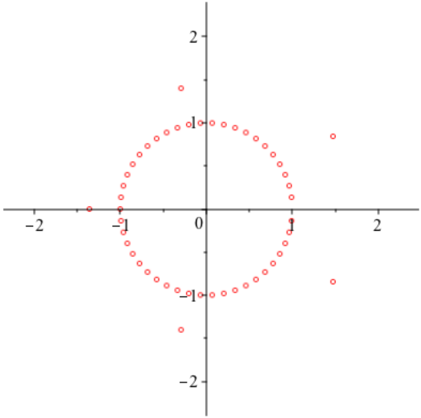


$(n, a, c) = (50, 3, 8)$ vs. $g(x) = (x^4 - x^3 + 8) \cdot (\sum_{j=0}^{46} x^j)$

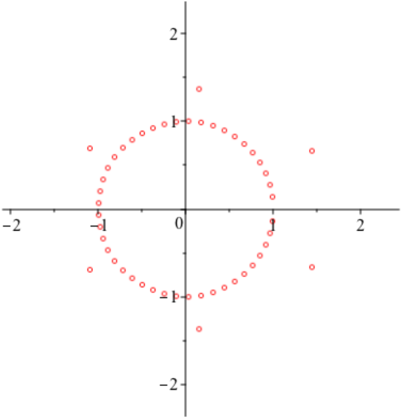


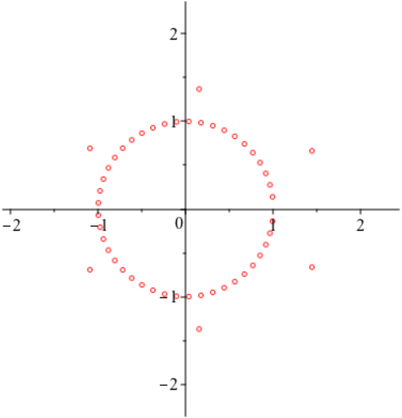
$(n, a, c) = (50, 4, 8)$ vs. $h(x) = (x^5 - x^4 + 8) \cdot (\sum_{j=0}^{45} x^j)$



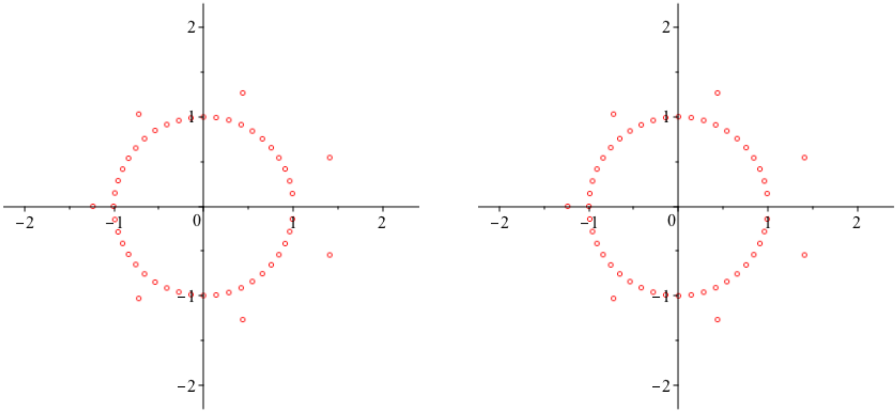


$(n, a, c) = (50, 5, 8)$ vs. $r(x) = (x^6 - x^5 + 8) \cdot (\sum_{j=0}^{44} x^j)$





$(n, a, c) = (50, 6, 8)$ vs. $s(x) = (x^7 - x^6 + 8) \cdot (\sum_{j=0}^{43} x^j)$



Overview

What Next?

- Negative values of c.

What Next?

- Negative values of c.
- Alternating coefficients; introduce a product of $(-1)^{j-1}$.

What Next?

- Negative values of c.
- Alternating coefficients; introduce a product of $(-1)^{j-1}$.
- I'm curious to know how certain parameters affect the "rate of convergence" in comparison to one-another.

Negative values of c

- Things are looking irreducible except for certain situations ($x^2 - a^2$ and $x^3 - a^3$).

Negative values of c

- Things are looking irreducible except for certain situations ($x^2 - a^2$ and $x^3 - a^3$).
- Roots have the same behaviour !

Alternating Coefficients

- Things are looking irreducible ... no exceptions this time.

Alternating Coefficients

- Things are looking irreducible ... no exceptions this time.
- Roots have “essentially” the same behaviour !

The End

Thank you very much for your time and patience ! Please feel free to ask any questions and I will do my best to answer them.