Implicit Representations of Epitrochoids and Hypotrochoids

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Definition 1

An implicit representation of an algebraic variety in $\mathbb{R}^n$ is of the form

$$\begin{aligned} F_1(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ F_m(x_1, \ldots, x_n) &= 0. \end{aligned}$$

Where the algebraic variety consists of the points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ which satisfy all of the above equations.

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$$\vdots$$
$$F_m(x_1, \ldots, x_n) = 0.$$

Where the algebraic variety consists of the points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ which satisfy all of the above equations.

If $m = n - 1$ then the algebraic variety is a curve.

Definition 2

A parametric representation of an algebraic variety in $\mathbb{R}^n$ is of the form

$$\begin{aligned} x_1 &= f_1(t_1, t_2, \ldots, t_m), \\ &\vdots \\ x_n &= f_n(t_1, t_2, \ldots, t_m). \end{aligned}$$

Where the algebraic variety consists of the points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which there exists $(t_1, t_2, \ldots, t_m)$ such that $(f_1(t_1, t_2, \ldots, t_m), \ldots, f_n(t_1, t_2, \ldots, t_m)) = (x_1, x_2, \ldots, x_n)$.

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A parametric representation of an algebraic variety in $\mathbb{R}^n$ is of the form

$$x_1 = f_1(t_1, t_2, \ldots, t_m),$$
$$\vdots$$
$$x_n = f_n(t_1, t_2, \ldots, t_m).$$

Where the algebraic variety consists of the points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which there exists $(t_1, t_2, \ldots, t_m)$ such that $(f_1(t_1, t_2, \ldots, t_m), \ldots, f_n(t_1, t_2, \ldots, t_m)) = (x_1, x_2, \ldots, x_n)$.

If $m = 1$ then the algebraic variety is a curve.

Definition 3

Converting from a parametric representation to an implicit one is called implicitization.

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Definition 4

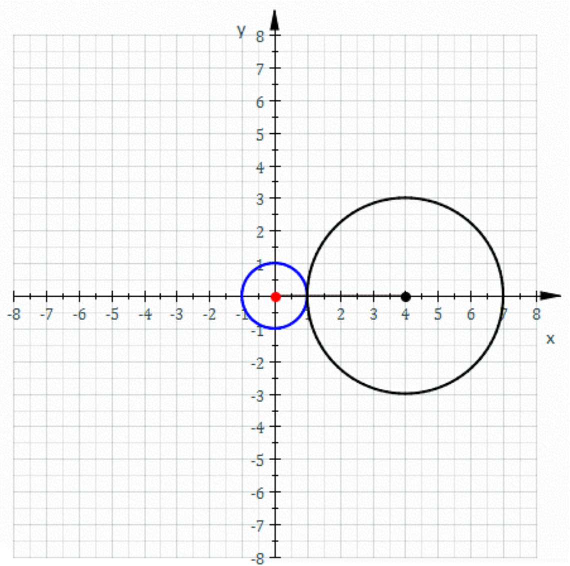
Converting from an implicit representation to a parametric one is called parametrization.

Definition 5

Epitrochoids are produced by tracing the path of a fixed point on a circle as it rolls along the outside of a fixed circle.

$d =$ distance from center of rolling circle to point that is tracing the curve
$R =$ radius of fixed circle at the origin
$r =$ radius of rolling circle
$\theta =$ angle between a line through the center of both circles and the x-axis

$$x(\theta) = (R+r)\cos\theta - d\cos\left(\frac{R+r}{r}\theta\right)$$
$$y(\theta) = (R+r)\sin\theta - d\sin\left(\frac{R+r}{r}\theta\right)$$


y
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-7
-6
-5
-4
-3
-2
-1
2
3
4
5
6
7
8
x

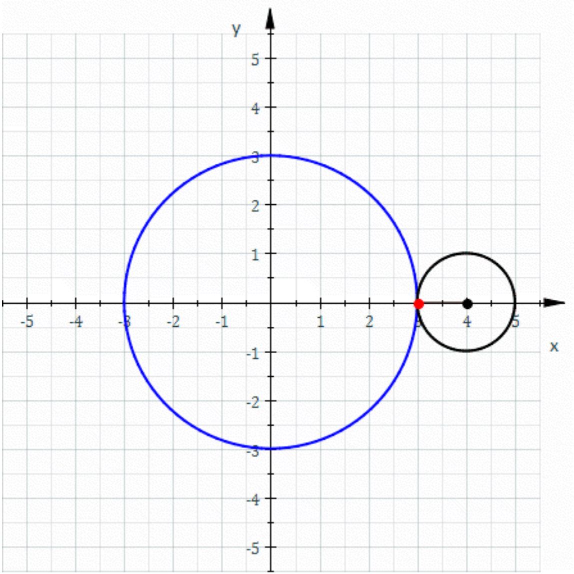
Definition 6

Epicycloids are produced when the distance from the fixed point on the moving circle to the center is equal to the radius of the moving circle.

$$x(\theta) = (R + r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right)$$
$$y(\theta) = (R + r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right)$$

For $k = R/r$, we also have the equivalent representation:

$$x(\theta) = r(k + 1)\cos\theta - r\cos((k + 1)\theta)$$
$$y(\theta) = r(k + 1)\sin\theta - r\sin((k + 1)\theta)$$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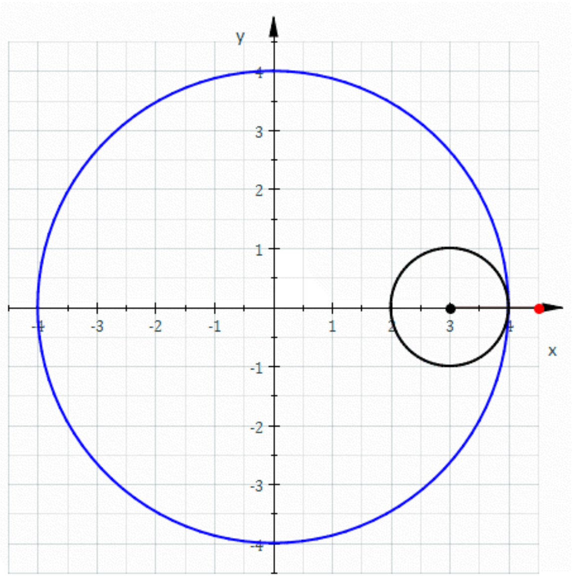
x

Definition 7

Hypotrochoids are produced by tracing the path of a fixed point on a circle as it rolls along the inside of a fixed circle.

$$x(\theta) = (R - r)\cos\theta + d\cos\left(\frac{R-r}{r}\theta\right)$$
$$y(\theta) = (R - r)\sin\theta - d\sin\left(\frac{R-r}{r}\theta\right)$$


y
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4
x

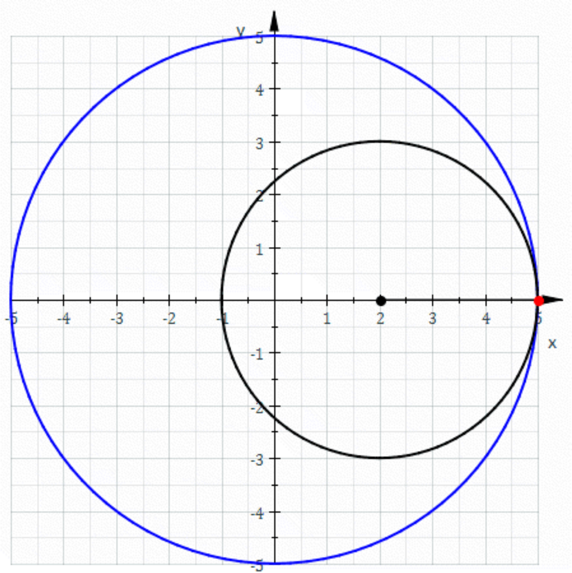
Definition 8

Hypocycloids are produced when the distance from the fixed point on the moving circle to the center is equal to the radius of the moving circle.

$$x(\theta) = (R - r)\cos\theta + r\cos\left(\frac{R - r}{r}\theta\right)$$
$$y(\theta) = (R - r)\sin\theta - r\sin\left(\frac{R - r}{r}\theta\right)$$

For $k = R/r$, we also have the equivalent representation:

$$x(\theta) = r(k - 1)\cos\theta + r\cos((k - 1)\theta)$$
$$y(\theta) = r(k - 1)\sin\theta - r\sin((k - 1)\theta)$$


y
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x

The Implicitization Problem

For rational parametric representations, there are many solutions.

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1. Sylvester resultant
2. Bézout resultant
3. Gröbner bases (omitted)

Let K be a field and $f, g \in K[x]$ such that,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$
$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0, \qquad b_m \neq 0.$$

The Sylvester resultant of f and g with respect to x, denoted $Res_x(f, g)$, is the following determinant:

$$
Res_x(f,g) = \begin{vmatrix}
a_n & a_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_0 & 0 & \cdots & 0 & 0 \\
0 & a_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_1 & a_0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & & & & & & & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & \cdots & \cdots & \cdots & \cdots & a_0 & 0 \\
0 & 0 & \cdots & 0 & a_n & a_{n-1} & a_{n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & a_1 & a_0 \\
b_m & b_{m-1} & \cdots & b_2 & b_1 & b_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
0 & b_m & \cdots & b_3 & b_2 & b_1 & b_0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & & & & & \ddots & & & & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & & & & & \ddots & & & & \vdots & \vdots \\
\vdots & \vdots & & & \ddots & & & & & \ddots & & & \vdots & \vdots \\
\vdots & \vdots & & & & \ddots & & & & & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & & & & \ddots & & & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & b_m & b_{m-1} & \cdots & \cdots & \cdots & b_0 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_m & \cdots & \cdots & \cdots & b_1 & b_0
\end{vmatrix}
$$

Example 9

$$f(x) = 7x^3 + 5x^2 + 3x + 1,$$
$$g(x) = 6x^2 + 4x + 2.$$

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$$Res_x(f,g) = \begin{vmatrix} 7 & 5 & 3 & 1 & 0 \\ 0 & 7 & 5 & 3 & 1 \\ 6 & 4 & 2 & 0 & 0 \\ 0 & 6 & 4 & 2 & 0 \\ 0 & 0 & 6 & 4 & 2 \end{vmatrix}$$

Theorem 10

Suppose we have a polynomial parametric representation of a curve in $\mathbb{R}^2$ *given by,*

$$x = x(t), \qquad y = y(t)$$

then the implicit representation is,

$$F(x, y) = Res_t(x - x(t), y - y(t)) = 0.$$

Sketch of Proof:

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1. Rewrite as

$$x - x(t) = \sum_{i=0}^{n}(a_i x - b_i)t^i = \sum_{i=0}^{n}\alpha_i t^i = 0$$

$$y - y(t) = \sum_{j=0}^{m}(c_j y - d_j)t^j = \sum_{j=0}^{m}\beta_i t^i = 0$$

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$$y - y(t) = \sum_{j=0}^{m}(c_j y - d_j)t^j = \sum_{j=0}^{m}\beta_i t^i = 0$$

2. Multiply through by $t^m, \ldots, t, 1$ and $t^n, \ldots, t, 1$, respectively.

Sketch of Proof:

1. Rewrite as

$$x - x(t) = \sum_{i=0}^{n} (a_i x - b_i) t^i = \sum_{i=0}^{n} \alpha_i t^i = 0$$

$$y - y(t) = \sum_{j=0}^{m} (c_j y - d_j) t^j = \sum_{j=0}^{m} \beta_i t^i = 0$$

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3. Arrange the system in the form $Ax = 0$

Sketch of Proof:

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$$x - x(t) = \sum_{i=0}^{n} (a_i x - b_i) t^i = \sum_{i=0}^{n} \alpha_i t^i = 0$$
$$y - y(t) = \sum_{j=0}^{m} (c_j y - d_j) t^j = \sum_{j=0}^{m} \beta_i t^i = 0$$

2. Multiply through by $t^m, \ldots, t, 1$ and $t^n, \ldots, t, 1$, respectively.
3. Arrange the system in the form $Ax = 0$
4. Nontrivial solution $\Leftrightarrow |A| = 0$

Theorem 11

Suppose we have a rational parametric representation of a curve in $\mathbb{R}^n$ given by:

$$x_1 = \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)},$$

$$\vdots$$

$$x_n = \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}.$$

Then an implicit representation can be found using Sylvester resultants.

Sketch of Proof:

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1. Rewrite as

$$
\begin{aligned}
y_1 &= x_1 \cdot g_1(t_1, \ldots, t_m) - f_1(t_1, \ldots, t_m) = 0, \\
&\vdots \\
y_n &= x_n \cdot g_n(t_1, \ldots, t_m) - f_n(t_1, \ldots, t_m) = 0, \\
y_{n+1} &= 1 - gy = 0.
\end{aligned}
$$

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2. Pair y_i's together (use one twice if necessary)

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2. Pair y_i's together (use one twice if necessary)
3. For each pair we have

$$y_i = \sum_{j=0}^{n_i} \alpha_{i,j} t_1^j, \qquad y_{i+1} = \sum_{j=0}^{n_{i+1}} \alpha_{i+1,j} t_1^j.$$

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Sketch of Proof:

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4. Apply theorem above for each pair
5. Repeat until all t_i's have been eliminated

Let K be a field and $f, g \in K[x]$ such that,

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, & a_n \neq 0, \\ g(x) &= b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0, & b_m \neq 0. \end{aligned}$$

WLoG, assume that $n \geq m$. Consider the following polynomial

$$P(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x - y} = \sum_{i,j=0}^{n-1} b_{i,j} x^i y^j$$

Then the Bézout resultant of f and g with respect to x is the following determinant:

$$Bez_x(f,g) = \begin{vmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} \\ b_{1,0} & b_{1,1} & \cdots & b_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-1} \end{vmatrix}$$

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$$Bez_x(f, g) = \begin{vmatrix} -4 & -8 & -4 \\ -2 & 7 & 5 \\ 6 & 4 & 2 \end{vmatrix}$$

Theorem 13

Suppose we have a polynomial parametric representation of a curve in $\mathbb{R}^2$ given by,

$$x = x(t), \quad y = y(t)$$

then the implicit representation is,

$$F(x, y) = Bez_t(x - x(t), y - y(t)) = 0.$$

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1. Let $f(x,t) = x - x(t) = 0$ and $g(y,t) = y - y(t) = 0$

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2. Consider the polynomial:

$$\begin{aligned} P(t,s) &= \frac{f(x,t)g(y,s) - f(x,s)g(y,t)}{t-s} \\ &= P_0(t) + P_1(t)s + P_2(t)s^2 + \cdots + P_{n-1}(t)s^{n-1} \end{aligned}$$

where $P_i(t) = \sum_{j=0}^{n-1} b_{i,j} t^j$

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   where $P_i(t) = \sum_{j=0}^{n-1} b_{i,j} t^j$
3. t is a solution $\Leftrightarrow$ $P(t,s) = 0$

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where $P_i(t) = \sum_{j=0}^{n-1} b_{i,j} t^j$

3. t is a solution $\Leftrightarrow P(t,s) = 0$
4.

$$\begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \cdots & b_{0,n-1} \\ b_{1,0} & b_{1,1} & b_{1,2} & \cdots & b_{1,n-1} \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & b_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1,0} & b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-1} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Theorem 14

Suppose we have a rational parametric representation of a curve in $\mathbb{R}^n$ given by:

$$x_1 = \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)},$$

$$\vdots$$

$$x_n = \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}.$$

Then an implicit representation can be found using Bézout resultants.

Sketch of Proof:

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1. Rewrite as

$$h_1(x_1, t_1, \ldots, t_m) = x_1 \cdot g_1(t_1, \ldots, t_m) - f_1(t_1, \ldots, t_m) = 0,$$
$$\vdots$$
$$h_n(x_n, t_1, \ldots, t_m) = x_n \cdot g_n(t_1, \ldots, t_m) - f_n(t_1, \ldots, t_m) = 0,$$
$$h_{n+1}(y, t_1, \ldots, t_m) = 1 - gy = 0.$$

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2. Pair h_i's together (use one twice if necessary)

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$$h_{n+1}(y, t_1, \ldots, t_m) = 1 - gy = 0.$$

2. Pair h_i's together (use one twice if necessary)
3. Consider the pair $\{h_i, h_{i+1}\}$. Let $n_{i,i+1} = \max\{\deg_{t_1}(h_i), \deg_{t_1}(h_{i+1})\}$.

$$P_{i,i+1}(t_1, s) = \frac{h_i(x_i, t_1, \ldots, t_m)h_{i+1}(x_{i+1}, s, \ldots, t_m) - h_i(x_i, s, \ldots, t_m)h_{i+1}(x_{i+1}, t_1, \ldots, t_m)}{t_1 - s}$$
$$= P_0(t_1) + P_1(t_1)s + \cdots + P_{n_{i,i+1}-1}(t_1)s^{n_{i,i+1}-1}$$

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$$\vdots$$
$$h_n(x_n, t_1, \ldots, t_m) = x_n \cdot g_n(t_1, \ldots, t_m) - f_n(t_1, \ldots, t_m) = 0,$$
$$h_{n+1}(y, t_1, \ldots, t_m) = 1 - gy = 0.$$

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$$P_{i,i+1}(t_1, s) = \frac{h_i(x_i, t_1, \ldots, t_m)h_{i+1}(x_{i+1}, s, \ldots, t_m) - h_i(x_i, s, \ldots, t_m)h_{i+1}(x_{i+1}, t_1, \ldots, t_m)}{t_1 - s}$$
$$= P_0(t_1) + P_1(t_1)s + \cdots + P_{n_{i,i+1}-1}(t_1)s^{n_{i,i+1}-1}$$

4. Apply theorem above for each pair

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1. Rewrite as
$$h_1(x_1, t_1, \dots, t_m) = x_1 \cdot g_1(t_1, \dots, t_m) - f_1(t_1, \dots, t_m) = 0,$$
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$$h_n(x_n, t_1, \dots, t_m) = x_n \cdot g_n(t_1, \dots, t_m) - f_n(t_1, \dots, t_m) = 0,$$
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$$P_{i,i+1}(t_1, s) = \frac{h_i(x_i, t_1, \dots, t_m)h_{i+1}(x_{i+1}, s, \dots, t_m) - h_i(x_i, s, \dots, t_m)h_{i+1}(x_{i+1}, t_1, \dots, t_m)}{t_1 - s}$$
$$= P_0(t_1) + P_1(t_1)s + \cdots + P_{n_{i,i+1}-1}(t_1)s^{n_{i,i+1}-1}$$
4. Apply theorem above for each pair
5. Repeat until all t_i's have been eliminated

Parametrization

Parametrization

1. No general solutions

Parametrization

1. No general solutions
2. A rational parametric representation may not exist
   1. Example 1: For $n \geq 3$

$$x^n + y^n = 1$$

   2. Example 2: (Most) Elliptic curves

$$x^3 + x^2y + xy^2 + y^3 + x^2 + xy + y^2 + x + y + 1 = 0$$

Let K be a field.

Definition 15

A relation $>$ on the set of monomials $x^\alpha, \alpha \in \mathbb{Z}^n_{\geq 0}$, or equivalently, a relation on $\mathbb{Z}^n_{\geq 0}$, is a monomial ordering on $K[x_1, \dots, x_n]$ if:

1. $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.
2. $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$ implies that $\alpha + \gamma > \beta + \gamma$.

Definition 16

Well-ordering is a total ordering for which every non-empty subset has a smallest element under that relation.

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Definition 17

X is totally ordered under $\leq$ if

1. $a \leq b$ and $b \leq a \Rightarrow a = b$
2. $a \leq b$ and $b \leq c \Rightarrow a \leq c$
3. $a \leq b$ or $b \leq a$

for all $a, b \in X$.

Definition 18

Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ be such that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. A lexicographical ordering is defined by the following condition: if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive, then we say that $\alpha >_{lex} \beta$. If $\alpha >_{lex} \beta$, then we write $x^\alpha >_{lex} x^\beta$.

Definition 18

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Proposition 1

A lexicographical ordering is a monomial ordering.

Definition 19

Let $>$ be a monomial order on $K[x_1, \ldots, x_n]$ and $f \in K[x_1, \ldots, x_n]$ be a nonzero polynomial where $f = \sum_\alpha a_\alpha x^\alpha$.

1. The multidegree of f, where the maximum is taken with respect to $>$, is defined by
$$\text{multideg}(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0).$$
2. From this we define the leading monomial of f to be
$$LM(f) = x^{\text{multideg}(f)}.$$
3. And the leading coefficient of f as
$$LC(f) = a_{\text{multideg}(f)} \in K.$$
4. Therefore, the leading term of f is
$$LT(f) = LC(f) \cdot LM(f).$$

Definition 20

Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal, $I \neq \{0\}$, and fix a monomial ordering on $K[x_1, \ldots, x_n]$. Then:

1. We denote the set of leading terms of nonzero elements of I by $LT(I)$. Hence,

$$LT(I) = \{cx^\alpha \mid \text{there exists } f \in I \setminus \{0\} \text{ with } LT(f) = cx^\alpha\}$$

2. We denote the ideal generated by the elements of $LT(I)$ by $\langle LT(I) \rangle$.

Definition 21

Let $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$. Then we define the polynomial ideal generated by these functions as

$$\langle f_1, \ldots, f_n \rangle = \{f \mid f = h_1 f_1 + \cdots + h_n f_n, h_i \in K[x_1, \ldots, x_n]\}$$

Definition 22

Fix a monomial order on the polynomial ring $K[x_1, \ldots, x_n]$. Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal such that $I \neq \{0\}$, a finite subset $G = \{g_1, \ldots, g_t\}$ of I is called a Gröbner basis if

$$I = \langle g_1, \ldots, g_t \rangle \text{ and } \langle LT(g_1), \ldots, LT(g_t) \rangle = \langle LT(I) \rangle.$$

Theorem 23

Fix a monomial ordering on $K[x_1, \ldots, x_n]$. If $I \subseteq K[x_1, \ldots, x_n]$ is an ideal, then I has a Gröbner basis.

Theorem 24

Given rational parametrization in an infinite field K:

$$x_1 = \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)},$$

$$\vdots$$

$$x_n = \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}.$$

- $g = g_1 \cdot g_2 \cdots g_n$
- $J = \langle g_1 x_1 - f_1, \ldots, g_n x_n - f_n, 1 - gy \rangle$
  $\subseteq K[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$
- *Lexicographical ordering: y and every t_i are greater than every x_i*

Theorem 24

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- $g = g_1 \cdot g_2 \cdots g_n$
- $J = \langle g_1 x_1 - f_1, \ldots, g_n x_n - f_n, 1 - gy \rangle$
  $\subseteq K[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$
- *Lexicographical ordering: y and every t_i are greater than every x_i*

Implicit form are the elements of the Gröbner basis of J not involving $y, t_1, \ldots, t_m$.

Relationship

1. Resultants: Total of $n! \cdot m!$
    1. Order for pairings
    2. Order of elimination

Relationship

1. Resultants: Total of $n! \cdot m!$
   1. Order for pairings
   2. Order of elimination
2. Gröbner: Total of $n! \cdot m!$
   1. Lexographical order
   2. Order of functions

Conjecture 1

Every implicit representation of an algebraic variety that can be found through resultants can also be found using the the Gröbner basis method of implicitization.

Extraneous Factors

Implicitization using resultants can result in extraneous factors that may be difficult to recognize and eliminate.

Extraneous Factors

Implicitization using resultants can result in extraneous factors that may be difficult to recognize and eliminate.

However, Gröbner bases do not generate extraneous factors.

Extraneous Factors

Example 25

A parametrization of the unit sphere is given by

$$x = \frac{1 - s^2 - t^2}{1 + s^2 + t^2},$$
$$y = \frac{2s}{1 + s^2 + t^2},$$
$$z = \frac{2t}{1 + s^2 + t^2}.$$

Using resultants: Pair (y_1, y_2) and (y_2, y_3) and eliminating s first:

$$(x^2 + y^2 + z^2 - 1)(x^2y^4 + y^6 + y^4z^2 - 4xy^2z^2 + 4x^2z^2 - y^4 - 4y^2z^2 + 8xz^2 + 4z^2) = 0.$$

The first factor is the implicit representation of the unit sphere.

Computational Complexity

1. Sylvester:
    - Determinant easy to construct
    - Large determinant can be costly

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Computational Complexity

1. Sylvester:
   - Determinant easy to construct
   - Large determinant can be costly
2. Bézout:
   - Smaller determinant
   - Division required can be costly
3. Gröbner:
   - Can also require a large amount of time or memory
   - Total degrees of intermediate polynomials can be quite large
   - Coefficients can be complicated rational numbers

Implicitization of Epitrochoids and Hypocycloids

Theorem 26

All trigonometric rational parametric representations of the form

$$x_1 = \frac{f_1(\sin(k_1\theta),\ldots,\sin(k_j\theta),\cos(k_1\theta),\ldots,\cos(k_j\theta),t_1,\ldots,t_m)}{g_1(\sin(k_1\theta),\ldots,\sin(k_j\theta),\cos(k_1\theta)\ldots,\cos(k_j\theta),t_1,\ldots,t_m)},$$

$$\vdots$$

$$x_n = \frac{f_n(\sin(k_1\theta),\ldots,\sin(k_j\theta),\cos(k_1\theta)\ldots,\cos(k_j\theta),t_1,\ldots,t_m)}{g_n(\sin(k_1\theta)\ldots,\sin(k_j\theta),\cos(k_1\theta)\ldots,\cos(k_j\theta),t_1,\ldots,t_m)}.$$

can be expressed as

$$x_1 = \frac{f_1(z,t_1,\ldots,t_m)}{g_1(z,t_1,\ldots,t_m)},$$

$$\vdots$$

$$x_n = \frac{f_n(z,t_1,\ldots,t_m)}{g_n(z,t_1,\ldots,t_m)}.$$

Sketch of Proof:

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1. $$\cos(k\theta) = \frac{e^{-ik\theta} + e^{ik\theta}}{2}, \qquad \sin(k\theta) = \frac{ie^{-ik\theta} - ie^{ik\theta}}{2}.$$

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2. $(k_1, \ldots, k_j) = (n_1/l_1, \ldots, n_j/l_j)$

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1. $$\cos(k\theta) = \frac{e^{-ik\theta} + e^{ik\theta}}{2}, \qquad \sin(k\theta) = \frac{ie^{-ik\theta} - ie^{ik\theta}}{2}.$$
2. $(k_1, \ldots, k_j) = (n_1/l_1, \ldots, n_j/l_j)$
3. let $\ell = lcm(l_1, \ldots, l_j)$

Sketch of Proof:

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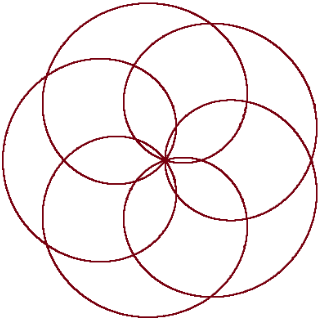
2. $(k_1, \ldots, k_j) = (n_1/l_1, \ldots, n_j/l_j)$
3. let $\ell = lcm(l_1, \ldots, l_j)$
4. let $z = e^{i\theta/\ell}$

Recall the parametric equations that describe an epitrochoid:

$$x(\theta) = (R+r)\cos\theta - d\cos\left(\frac{R+r}{r}\theta\right)$$
$$y(\theta) = (R+r)\sin\theta - d\sin\left(\frac{R+r}{r}\theta\right)$$



Theorem 27

Epitrochoids are rational parametric curves and, hence, have a representation that can be used for implicitization.

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Epitrochoids are rational parametric curves and, hence, have a representation that can be used for implicitization.

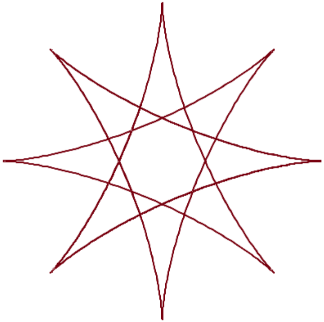
Sketch of Proof:

1. Use the above theorem with $z = e^{i\theta/r}$
2. This will give us:

$$0 = dz^{2R+2r} - (R+r)z^{R+2r} + 2xz^{R+r} - (R+r)z^{R} + d$$
$$0 = dz^{2R+2r} - (R+r)z^{R+2r} - 2iyz^{R+r} + (R+r)z^{R} - d$$

Recall the parametric equations that describe a hypotrochoid:

$$x(\theta) = (R - r)\cos\theta + d\cos\left(\frac{R-r}{r}\theta\right)$$
$$y(\theta) = (R - r)\sin\theta - d\sin\left(\frac{R-r}{r}\theta\right)$$



Theorem 28

Hypotrochoids are rational parametric curves and, hence, have a representation that can be used for implicitization.

Theorem 28

Hypotrochoids are rational parametric curves and, hence, have a representation that can be used for implicitization.

Sketch of Proof:

1. Use the above theorem with $z = e^{i\theta/r}$
2. 5 Cases:
    1. $k < 1$
    2. $k = 1$
    3. $1 < k < 2$
    4. $k = 2$
    5. $k > 2$

Case 2 and 4

$k = 1$:

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$$x(\theta) = d, \qquad y(\theta) = 0.$$

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$k = 2$:

Case 2 and 4

$k = 1$:

$$x(\theta) = d, \qquad y(\theta) = 0.$$

$k = 2$:

$$x(\theta) = (r + d)\cos\theta$$
$$y(\theta) = (r - d)\sin\theta$$

Case 1, 3, and 5

Case 1, 3, and 5

$k < 1$:

$$0 = (R-r)z^{2r} - 2xz^{r} + dz^{2r-R} + dz^{R} + (R-r)$$
$$0 = (R-r)z^{2r} + 2iyz^{r} + dz^{2r-R} - dz^{R} - (R-r)$$

$1 < k < 2$:

$$0 = (R-r)z^{2r} + dz^{R} - 2xz^{r} + dz^{2r-R} + (R-r)$$
$$0 = (R-r)z^{2r} - dz^{R} + 2iyz^{r} + dz^{2r-R} - (R-r)$$

$k > 2$:

$$0 = dz^{2R-2r} + (R-r)z^{R} - 2xz^{R-r} + (R-r)z^{R-2r} + d$$
$$0 = dz^{2R-2r} - (R-r)z^{R} - 2iyz^{R-r} + (R-r)z^{R-2r} - d$$

Theorem 29

If $k < 1$, then the hypotrochoid produced will be an epitrochoid.

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Theorem 30

If $k_1 > 2$, then there exists k_2 where $1 < k_2 < 2$ such that k_2 produces the same hypotrochoid as k_1.

Theorem 29

If $k < 1$, then the hypotrochoid produced will be an epitrochoid.

Theorem 30

If $k_1 > 2$, then there exists k_2 where $1 < k_2 < 2$ such that k_2 produces the same hypotrochoid as k_1.

Sketch of Proofs:

- Simplify after appropriate substitutions

Notation

- m = radius of fixed circle
- n = radius of rolling circle

Notation

- $m =$ radius of fixed circle
- $n =$ radius of rolling circle
- $k = m/n = R/r$ where $gcd(R, r) = 1$

Theorem 31

The following parametric representation can be used to implicitize an epicycloid with Gröbner bases and resultants:

$$0 = anz^{2m+2n} - a(m+n)z^{m+2n} + 2nxz^{m+n} - a(m+n)z^{m} + an,$$
$$0 = anz^{2m+2n} - a(m+n)z^{m+2n} - 2niyz^{m+n} + a(m+n)z^{m} - an.$$

Theorem 31

The following parametric representation can be used to implicitize an epicycloid with Gröbner bases and resultants:

$$0 = anz^{2m+2n} - a(m+n)z^{m+2n} + 2nxz^{m+n} - a(m+n)z^{m} + an,$$
$$0 = anz^{2m+2n} - a(m+n)z^{m+2n} - 2niyz^{m+n} + a(m+n)z^{m} - an.$$

Sketch of Proof:

1. Use above theorem with new notation
2. Multiply through by n and selectively let $a = n$

Theorem 32

The following parametric representation can be used to implicitize a hypocycloid with Gröbner bases and resultants:

$k < 1$:

$$0 = a(m-n)z^{2n} - 2nxz^n + anz^{2n-m} + anz^m + a(m-n),$$
$$0 = a(m-n)z^{2n} + 2niyz^n + anz^{2n-m} - anz^m - a(m-n).$$

$1 < k < 2$:

$$0 = a(m-n)z^{2n} + anz^m - 2nxz^n + anz^{2n-m} + a(m-n),$$
$$0 = a(m-n)z^{2n} - anz^m + 2niyz^n + anz^{2n-m} - a(m-n).$$

$k > 2$:

$$0 = anz^{2m-2n} + a(m-n)z^m - 2nxz^{m-n} + a(m-n)z^{m-2n} + an,$$
$$0 = anz^{2m-2n} - a(m-n)z^m - 2niyz^{m-n} + a(m-n)z^{m-2n} - an.$$

Conjecture 2

The implicit representation of an epicycloid where R is odd is of the form:

$$F(x,y) = \sum_{i=0}^{R+r} p_{(2R+2r-2i)}(m,n) n^{2R+2r-2i}(x^2+y^2)^i + n^{R+2r} \sum_{i=0}^{(R-1)/2} p_{(R+2r,i)}(m,n) x^{R-2i} y^{2i} \qquad (1)$$

where $p_{(2R+2r-2i)}(m,n)$ and $p_{(R+2r,j)}(m,n)$ are polynomials in terms of m and n.

Before we describe these polynomials, we require the following function:

$$G(n) = \begin{cases} n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

If $R+2r < j \leq 2R+2r$, then $j = 2R-2r-2i$ for some $i \in \mathbb{Z}$ and

$$p_{(j)}(m,n) = m^{G(R-2i)} n^{2i} (m+2n)^{G(R-2i)} \sum_{h=0}^{2i} \alpha_h m^{2i-h} n^h.$$

For $0 \leq i \leq R/2$, the coefficient of $x^{R-2i} y^{2i}$ is

$$p_{(R+2r,i)}(m,n) = (-1)^{1+i} \binom{R}{2i} 2n^{R-r} (m+n)^{R+r}.$$

If $\lfloor R/2+2r \rceil_{2,1} \leq j < R+2r$, where $\lfloor \cdot \rceil_{2,1}$ indicates to round to the closest even integer, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_{(j)}(m,n) = n^{G(2R-d)} \sum_{h=0}^{2+d} \beta_h m^{(2+d)-h} n^h.$$

where $d = j - \lfloor R/2+2r \rceil_{2,1}$. If $0 \leq j < \lfloor R/2+2r \rceil_{2,1}$, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_j(m,n) = (-1)^r n^{2R-j} \sum_{h=0}^{j} \gamma_h m^{j-h} n^h.$$

Example 33

Epicycloid: $m = 3,\ n = 2$

$$
\begin{aligned}
F(x, y) = {} & p_{10} a^{10} + p_8 a^8 (x^2 + y^2) + a^7 (p_{7,0} x^3 + p_{7,1} x y^2) \\
& + p_6 a^6 (x^2 + y^2)^2 + p_4 a^4 (x^2 + y^2)^3 + p_2 a^2 (x^2 + y^2)^4 \\
& + p_0 (x^2 + y^2)^5
\end{aligned}
$$

$p_{10} = m^3 (m + 2n)^3$
$p_8 = mn^2 (m + 2n)(2m^2 + 4mn - 3n^2)$
$p_{7,0} = -2n(m + n)^5$
$p_{7,1} = 6n(m + n)^5$
$p_6 = n^4 (3m^2 + 6mn - 7n^2)$
$p_4 = n^2 (m^4 + 4m^3 n + 9m^2 n^2 + 10mn^3 + 14n^4)$
$p_2 = -n^4 (2m^2 + 4mn + 7n^2)$
$p_0 = n^6$

Conjecture 3

The implicit representation of an epicycloid where R is even is of the form:

$$F(x,y) = \sum_{\substack{i=0 \\ i \neq R/2}}^{R+r} p_{(2R+2r-2i)}(m,n) n^{2R+2r-2i} (x^2+y^2)^i + n^{R+2r} \sum_{i=0}^{R/2} p_{(R+2r,i)}(m,n) x^{R-2i} y^{2i} \quad (2)$$

where $p_{(2R+2r-2i)}(m,n)$ *and* $p_{(R+2r,j)}(m,n)$ *are polynomials in terms on m and n.*

If $R+2r<j\leq 2R+2r$, then $j=2R+2r-2i$ for some $i\in\mathbb{Z}$ and

$$p_{(j)}(m,n)=m^{G(R-2i)}n^{2i}(m+2n)^{G(R-2i)}\sum_{h=0}^{2i}\alpha_h m^{2i-h}n^h$$

where G is the function defined above. For $0\leq i\leq R/2$, the coefficient of $x^{R-2i}y^{2i}$ is

$$p_{(R+2r,i)}(m,n)=(-1)^{i+1}n^{R-r}\sum_{h=0}^{R+r}\beta_{h,i}m^{(R+r)-h}n^h.$$

If $\lfloor R/2 + 2r \rfloor_{2,2} \leq j < R + 2r$, where $\lfloor \cdot \rfloor_{2,2}$ indicates to round down to the closest even integer, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_{(j)}(m, n) = \sum_{h=0}^{i} \beta_h m^{i-h} n^h.$$

Lastly, if $0 \leq j < \lfloor R/2 + 2r \rfloor_{2,2}$, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_j(m, n) = (-1)^{r+i} n^{2R-j} \sum_{h=0}^{j} \gamma_h m^{j-h} n^h.$$

Example 34

Epicycloid: $m = 4,\ n = 1$

$$F(x,y) = p_{10}a^{10} + p_8 a^8(x^2+y^2) + a^6(p_{6,0}x^4 + p_{6,1}x^2y^2 + p_{6,0}y^4) + p_4a^4(x^2+y^2)^3 + p_2a^2(x^2+y^2)^4 + p_0(x^2+y^2)^5$$

$$p_{10} = m^4(m+2n)^4$$

$$p_8 = m^2n^2(m+2n)^2(m^2+2mn-4n^2)$$

$$p_{6,0} = -mn^3(2m^4+9m^3n+16m^2n^2+23mn^3+24n^4)$$

$$p_{6,1} = 2n^3(6m^5+31m^4n+64m^3n^2+57m^2n^3+16mn^4+8n^5)$$

$$p_4 = n^6(m^2+2mn-9n^2)$$

$$p_2 = n^6(m^2+2mn+6n^2)$$

$$p_0 = -n^8$$

Conjecture 4

The implicit representation of a hypocycloid where R is odd is of the form:

$$k < 2: \quad F(x,y) = \sum_{i=0}^{r} p_{(2r-2i)}(m,n) n^{2r-2i}(x^2+y^2)^i + n^{2r-R} \sum_{i=0}^{(R-1)/2} p_{(2r-R,i)}(m,n) x^{R-2i} y^{2i}$$

$$k > 2: \quad F(x,y) = \sum_{i=0}^{R-r} p_{(2R-2r-2i)}(m,n) n^{2R-2r-2i}(x^2+y^2)^i + n^{R-2r} \sum_{i=0}^{(R-1)/2} p_{(R-2r,i)}(m,n) x^{R-2i} y^{2i}$$

$k < 2$

If $2R - 4r - 2 < j \leq 2R - 2r$, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_{(j)}(m,n) = (-1)^i m^{G(R-2i)} n^{2i} (m-2n)^{G(R-2i)} \sum_{h=0}^{2i} \alpha_h m^{2i-h} n^h.$$

If $R - 2r < j \leq 2R - 4r - 2$, then

$$p_{(j)}(m,n) = (-1)^i m^{G(R-2i)} n^{G(2R-j)} (m-2n)^{G(R-2i)} \sum_{h=0}^{2+d} \beta_h m^{(2+d)-h} n^h.$$

where $d = (2R - 4r - 2) - j$. For $0 \leq i \leq (R-1)/2$, the coefficient of $x^{R-2i} y^{2i}$ is

$$p_{(|R-2r|,i)}(m,n) = (-1)^{R-r} \binom{R}{2i} 2n^{R+r} (m-n)^{R-r}.$$

Lastly, if $0 \leq j \leq R - 2r$, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_j(m,n) = (-1)^r n^{2R-j} \sum_{h=0}^{j} \gamma_h m^{j-h} n^h.$$

Example 35

Hypocycloid: $m = 5,\ n = 2$

$$F(x,y) = p_6 a^6 + p_4 a^4(x^2+y^2) + p_2 a^2 (x^2+y^2)^2 + a(p_{1,0}x^5 + p_{1,1}x^3y^2 + p_{1,2}xy^4) + p_0(x^2+y^2)^3$$

$$p_6 = m^5(m-2n)^5$$

$$p_4 = -m^3n^2(m-2n)^3(2m^2-4mn+5n^2)$$

$$p_2 = mn^4(m-2n)(m^4-4m^3n+7m^2n^2-6mn^3+5n^4)$$

$$p_{1,0} = -2n^7(m-n)^3$$

$$p_{1,1} = 20n^7(m-n)^3$$

$$p_{1,2} = -10n^7(m-n)^3$$

$$p_0 = n^{10}$$

Conjecture 5

The implicit representation of a hypocycloid where R is even is of the form:

$$k < 2: \quad F(x,y) = \sum_{\substack{i=0 \\ i \neq R/2}}^{r} p_{(2r-2i)}(m,n) n^{2r-2i} (x^2+y^2)^i + n^{2r-R} \sum_{i=0}^{R/2} p_{(2r-R,i)}(m,n) x^{R-2i} y^{2i}$$

$$k > 2: \quad F(x,y) = \sum_{\substack{i=0 \\ i \neq R/2}}^{R-r} p_{(2R-2r-2i)}(m,n) n^{2R-2r-2i} (x^2+y^2)^i + n^{R-2r} \sum_{i=0}^{R/2} p_{(R-2r,i)}(m,n) x^{R-2i} y^{2i}$$

$k < 2$

If $2R - 4r - 2 < j \leq 2R - 2r$, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_{(j)}(m, n) = (-1)^i m^{G(R-2i)} n^{2i} (m - 2n)^{G(R-2i)} \sum_{h=0}^{2i} \alpha_h m^{2i-h} n^h.$$

If $R - 2r < j \leq 2R - 4r - 2$, then

$$p_{(j)}(m, n) = (-1)^r m^{G(R-2i)} n^{G(2R-j)} (m - 2n)^{G(R-2i)} \sum_{h=0}^{2+d} \beta_h m^{(2+d)-h} n^h.$$

where $d = (2R - 4r - 2) - j$. For $0 \leq i \leq R/2$, the coefficient of $x^{R-2i} y^{2i}$ is

$$p_{(|R-2r|,i)}(m, n) = (-1)^i n^{R+r} \sum_{h=0}^{R-r} \beta_{h,i} m^{(R-r)-h} n^h.$$

Lastly, if $0 \leq j \leq R - 2r$, then $j = 2R - 2r - 2i$ for some $i \in \mathbb{Z}$ and

$$p_j(m, n) = (-1)^r n^{2R-j} \sum_{h=0}^{j} \gamma_h m^{j-h} n^h.$$

Example 36

Hypocycloid: $m = 4,\ n = 1$

$$F(x,y) = p_6 a^6 + p_4 a^4 (x^2 + y^2) + a^2 (p_{2,0} x^4 + p_{2,1} x^2 y^2 + p_{2,1} y^4) + p_0 (x^2 + y^2)^3$$

$$p_6 = -m^4 (m - 2n)^4$$

$$p_4 = m^2 n^2 (m - 2n)^2 (m^2 - 2mn + 4n^2)$$

$$p_{2,0} = -mn^5 (2m^2 - 7mn + 8n^2)$$

$$p_{2,1} = 2n^5 (6m^3 - 17m^2 n + 16mn^2 - 8n^3)$$

$$p_0 = n^8$$

Definition 37

Let $f : V \to W$ be a function between vector spaces over a field K and b be an integer. We say that f is homogeneous of degree b if

$$f(\alpha \mathbf{v}) = \alpha^b f(\mathbf{v})$$

for all nonzero $\alpha \in K$ and $\mathbf{v} \in V$.

Definition 37

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$$f(\alpha \mathbf{v}) = \alpha^b f(\mathbf{v})$$

for all nonzero $\alpha \in K$ and $\mathbf{v} \in V$.

Example 38

The function

$$f(x, y, z) = x^3 + 5x^2y + zy^2$$

is homogeneous of degree 3.

Corollary 39

Suppose we have an epicycloid (or hypocycloid) with $k = R/r$ given by:

$$x_1 = r(k \pm 1)\cos\theta \mp r\cos((k \pm 1)\theta),$$
$$y_1 = r(k \pm 1)\sin\theta - r\sin((k \pm 1)\theta).$$

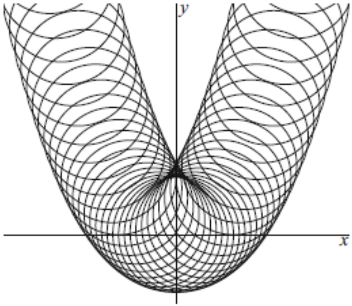
With implicit representation,

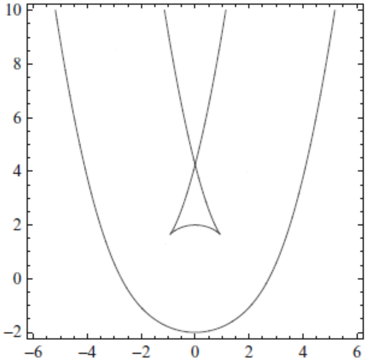
$$F_1(x_1, y_1) = \sum_{i,j,\kappa,\lambda} p_\lambda(R, r) r^i x_1^j y_1^\kappa = 0.$$

Then another epicycloid (or hypocycloid) with $k = m/n$ has an implicit representation of the form:

$$F_2(x_2, y_2) = \sum_{i,j,\kappa,\lambda} p_\lambda(m, n) n^i x_2^j y_2^\kappa = 0.$$

Suppose we have a family of curves, such as a series of circles or lines. The envelope of this family is a curve that is tangent to the curves in this family.





Definition 40

Let $F : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ be a smooth map. Express the coordinates on the left as $(t, x_1, \ldots, x_r)$; we view F as a family of functions of x, parametrized by t. Denote $F_t : \mathbb{R}^r \to \mathbb{R}$ for the functions $F_t(x) = F(t, x)$. Then the family of curves is determined by F and consists of $V(F_t)$, the varieties of F_t as t varies over $\mathbb{R}$.

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Definition 41

The set

$$\mathscr{D}_F = \{\mathbf{x} \in \mathbb{R}^r : \text{there exists } t \in \mathbb{R} \text{ such that } F(t, \mathbf{x}) = \partial F/\partial t(t, \mathbf{x}) = 0\}$$

is the envelope of the family F.

Theorem 42

The envelope of the family of curves given by

$$F(t,x) = x_1(\sin(mt)+\sin(nt)) - x_2(\cos(nt)+\cos(mt)) - \sin(t(m-n))$$

where $m, n \in \mathbb{Z}_{>0}$ *is an epicycloid with* $k = (m-n)/n$.

Sketch of Proof:

1. Use the definitions
2. Apply a uniform scaling of a uniform scaling factor of $m+n$
3. Let $R = (m-n)$ and $r = n$

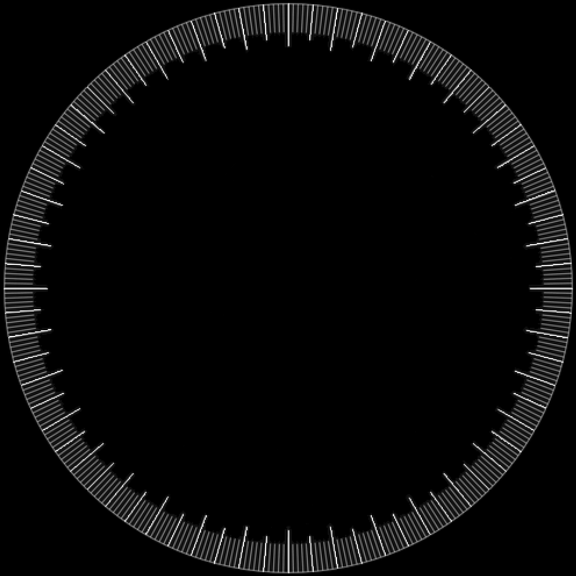
Since $F(x,t)=0$, we have that

$$x_2 = \left(\frac{\sin(nt)+\sin(mt)}{\cos(nt)+\cos(mt)}\right)x_1 + \frac{-\sin(t(m-n))}{\cos(nt)+\cos(mt)}.$$

Since $F(x,t)=0$, we have that

$$x_2 = \left(\frac{\sin(nt)+\sin(mt)}{\cos(nt)+\cos(mt)}\right)x_1 + \frac{-\sin(t(m-n))}{\cos(nt)+\cos(mt)}.$$

Therefore, the family of curves is a series of lines which pass through the points $(\cos(nt),\sin(nt))$ and $(-\cos(mt),-\sin(mt))$.



Theorem 43

The envelope of the family of curves given by

$$F(t,x) = x_1(\sin(nt) - \sin(mt)) - x_2(\cos(mt) + \cos(nt)) + \sin(t(m+n))$$

where $m, n \in \mathbb{Z}_{>0}$ *with* $m \neq n$ *is a hypocycloid with* $k = (m+n)/n$.

Sketch of Proof:

1. Use the definitions
2. Apply a uniform scaling of a uniform scaling factor of $m - n$
3. Let $R = (m+n)$ and $r = n$

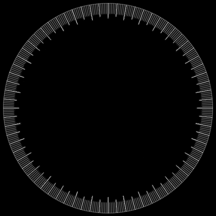
Since $F(x,t)=0$, we have that

$$x_2 = \left(\frac{\sin(nt)+\sin(mt)}{\cos(nt)-\cos(mt)}\right)x_1 + \frac{-\sin(t(m+n))}{\cos(nt)-\cos(mt)}.$$

Since $F(x, t) = 0$, we have that

$$x_2 = \left(\frac{\sin(nt) + \sin(mt)}{\cos(nt) - \cos(mt)} \right) x_1 + \frac{-\sin(t(m+n))}{\cos(nt) - \cos(mt)}.$$

Therefore, the family of curves is a series of lines which pass through the points $(\cos(nt), \sin(nt))$ and $(\cos(mt), -\sin(mt))$.



Thank You!