# Multiple zeta values for classical special functions

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# MZV

Given some function $G(z)$, we denote by $\{a_n\}$ the set of its zeros (assumed to be non-zero complex numbers).
Then define a **zeta function** associated with $G$ as

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}.$$

# MZV

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Then define a **zeta function** associated with $G$ as

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}.$$

We also construct a **multiple zeta function** as

$$\zeta_G(s_1, \dots, s_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{a_{n_1}^{s_1} \cdots a_{n_r}^{s_r}}$$

and a **multiple zeta-starred function** as

$$\zeta_G^*(s_1, \dots, s_r) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{a_{n_1}^{s_1} \cdots a_{n_r}^{s_r}}.$$

# MZV

Denote

$$\zeta_G\left(\{s\}^{\,n}\right) = \zeta_G\left(\{s, s, \ldots\}\right)$$
$$\zeta_G^*\left(\{s\}\}^n\right) = \zeta_G^*\left(\{s, s, \ldots\}\right)$$

# MZV and ground states

Given a quantum system with eigenvalues $\{E_1, E_2, \ldots\}$, arranged in decreasing order of magnitude, the quantum zeta function $Z(s)$ is defined as the series

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{E_n^s},$$

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$$E_1 \sim Z\left(s\right)^{\frac{1}{s}}$$

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In the case of a quantum multiple zeta function $Z\left(s_1, s_2, \ldots, s_r\right)$, an equivalent is

$$Z\left(\{s\}^{r}\right)^{\frac{1}{s}} \sim \frac{1}{E_1 E_2 \cdots E_r}$$

# outline

We address two questions in this talk:

- what does the knowledge of the **Weierstrass product factorization** of the function $G$ tell us about the MZV built on its zeros ?

# outline

We address two questions in this talk:

- what does the knowledge of the **Weierstrass product factorization** of the function $G$ tell us about the MZV built on its zeros ?
- what are the benefits of introducing **generalized Bernoulli numbers** in the computation of the MZV's ?

# Definitions and Methodology

Symmetric functions
- the elementary symmetric functions $e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$,
- the complete symmetric functions $h_r = \sum_{i_1 \leq i_2 \leq \ldots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$
- the power sums $p_r = \sum_i x_i^r$.

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- the power sums $p_r = \sum_i x_i^r$.

The symmetric functions have generating functions

$$E(t) = \sum_{k=0}^{\infty} e_k t^k = \prod_{i=1}^{\infty} (1 + t x_i)\,,$$

$$H(t) = \sum_{k=0}^{\infty} h_k t^k = \prod_{i=1}^{\infty} (1 - t x_i)^{-1} = E(-t)^{-1},$$

$$P(t) = \sum_{k=1}^{\infty} p_k t^{k-1} = \sum_{i=1}^{\infty} \frac{x_i}{1 - t x_i} = \frac{H'(t)}{H(t)} = \frac{E'(-t)}{E(-t)}.$$

# Definitions and Methodology

## Lemma

*When* $x_i = \frac{1}{a_i^s}$,

$$p_k = \zeta_G\left(ks\right) = \sum_{n=1}^{\infty} \frac{1}{a_n^{ks}}$$

$$e_k = \zeta_G\left(\{s\}^k\right) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{a_{n_1}^s a_{n_2}^s \ldots a_{n_k}^s}$$

*and*

$$h_k = \zeta_G^*\left(\{s\}^k\right) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{a_{n_1}^s a_{n_2}^s \ldots a_{n_k}^s}$$

# Definitions and Methodology

As a consequence define the averaged zeta

$$S(mn,k) = \sum_{|\mathbf{a}|=n} \zeta_G(ma_1,\ldots,ma_k),$$

$$S_G^*(mn,k) = \sum_{|\mathbf{a}|=n} \zeta_G^*(ma_1,\ldots,ma_k)$$

# Definitions and Methodology

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$$S(mn,k) = \sum_{|\mathbf{a}|=n} \zeta_G(ma_1,\ldots,ma_k),$$

$$S^*_G(mn,k) = \sum_{|\mathbf{a}|=n} \zeta^*_G(ma_1,\ldots,ma_k)$$

The following generating products hold:

$$\prod_{k=1}^{\infty} \frac{\left(1+\frac{(y-1)t}{a_k^s}\right)}{\left(1-\frac{t}{a_k^s}\right)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_G(sn,k)y^k t^n,$$

$$\prod_{k=1}^{\infty} \frac{\left(1-\frac{t}{a_k^s}\right)}{\left(1-\frac{(y+1)t}{a_k^s}\right)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S^*_G(sn,k)y^k t^n$$

# Hypergeometric Zeta

The **hypergeometric zeta function** is defined by

$$\zeta_{a,b}(s) = \sum_{k \geq 1} \frac{1}{z_{a,b;k}^{s}},$$

where $z_{a,b;k}$ are the zeros of the classic Kummer (confluent hypergeometric) function

$$\Phi_{a,b}(z) := {}_1F_1\left( \begin{matrix} a \\ a+b \end{matrix} ; z \right).$$

The zeros $z_{a,b;k}$ are pair-wise complex conjugated.

# Hypergeometric Zeta

For $a = 1, b = 0$, we have

$$\Phi_{1,0}(z) = {}_1F_1\left(\begin{array}{c} 1 \\ 1 \end{array}; z\right) = e^z.$$

For $b = a$, we have

$$\Phi_{a,a}(z) = {}_1F_1\left(\begin{array}{c} a \\ 2a \end{array}; z\right) = e^{\frac{z}{2}} 2^{2a-1} \Gamma\left(a + \frac{1}{2}\right) \frac{I_{a-\frac{1}{2}}(z)}{z^{a-\frac{1}{2}}}.$$

# Hypergeometric Zeta

This function has the Weierstrass factorization

$$ {}_1F_1\left( \begin{matrix} a \\ a+b \end{matrix} ; z \right) = e^{\frac{a}{a+b} z} \prod_{k \geq 1} \left( 1 - \frac{z}{z_{a,b;k}} \right) e^{\frac{z}{z_{a,b;k}}}. $$

# Hypergeometric Zeta

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We deduce a (nontrivial) generating function for $\zeta_{a,b}(s)$ as

$$ \sum_{k=1}^{\infty} \zeta_{a,b}(k+1) z^k = \frac{b}{a+b} \left( \frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} - 1 \right). $$

# Hypergeometric Zeta

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Moreover, from the Weierstrass factorization we deduce

$$ {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; \imath z\right) {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; -\imath z\right) = \prod_{k\geq 1}\left(1+\frac{z^2}{z_{a,b;k}^2}\right). $$

We need an identity originally due to Ramanujan [1, Entry 18, p.61] and later proved by Preece.

# Hypergeometric Zeta

## Lemma

*We have the identity*

$$ {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix};\imath z\right){}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix};-\imath z\right) = {}_2F_3\left(\begin{matrix} a,b \\ a+b,\frac{a+b}{2},\frac{a+b+1}{2} \end{matrix};-\frac{z^2}{4}\right). $$

*The multiple zeta value* $\zeta_{a,b}\left(\{2\}^n\right)$ *can be deduced as*

$$ \zeta_{a,b}\left(\{2\}^n\right) = \frac{(-1)^n}{n!}\frac{(a)_n\,(b)_n}{(a+b)_n\,(a+b)_{2n}}. $$

*For example,*

$$ \zeta_{a,b}\left(\{2\}\right) = -\frac{ab}{(a+b)^2\,(a+b+1)} $$

# Hypergeometric Zeta

The values of $\zeta_{a,b}(\{2r\}^n)$ for $r = 1, 2, \ldots$ can be recursively computed from $\zeta_{a,b}(\{2\}^n)$ using the following general result.

## Theorem

*Take $m \in \mathbb{Z}$, $m > 0$ and $\omega$ a primitive $m-$th root of unity. Then*

$$\zeta_G(\{ms\}^n) = (-1)^{n(m-1)} \sum_{l_0+l_1+\cdots+l_{m-1}=mn} \prod_{j=0}^{m-1} \zeta_G(\{s\}^{l_j})\omega^{jl_j}.$$

# Hypergeometric Zeta

## Proof.

Start with the identity $1 - z^m = \prod_{j=0}^{m-1}(1 - z\omega^j)$ for $m > 0$, so that

$$\sum_{n=0}^{\infty}(-1)^n \zeta_G(\{ms\}^n)t^{mn} = \prod_{k=1}^{\infty}\left(1 - \frac{t^m}{a_k^{ms}}\right) = \prod_{k=1}^{\infty}\prod_{j=0}^{m-1}\left(1 - \frac{t\omega^j}{a_k^s}\right)$$

$$= \prod_{j=0}^{m-1}\prod_{k=1}^{\infty}\left(1 - \frac{t\omega^j}{a_k^s}\right) = \prod_{j=0}^{m-1}\left(\sum_{n=0}^{\infty}\omega^{jn}(-1)^n\zeta_G(\{s\}^n)t^n\right).$$

□

# Hypergeometric Zeta

## Theorem

*The multiple zeta value $\zeta_{a,b}(\{4\}^n)$ is equal to*

$$\zeta_{a,b}(\{4\}^n) = \frac{(-1)^n}{(2n)!} \frac{(a)_{2n}(b)_{2n}}{(a+b)_{2n}(a+b)_{4n}}$$

$$\times {}_6F_5\left( \begin{matrix} -2n, 1-2n-a-b, 1-2n-\frac{a+b}{2}, 1-2n-\frac{a+b+1}{2}, a, b \\ 1-2n-a, 1-2n-b, a+b, \frac{a+b}{2}, \frac{a+b+1}{2} \end{matrix} ; -1 \right)$$

*For example,*

$$\zeta_{a,b}(\{4\}) = -\frac{ab(a^3+a^2(1-2b)+b^2(1+b)-2ab(2+b)}{(a+b)^4(1+a+b)^2(2+a+b)(3+a+b)}.$$

# Hypergeometric Zeta

## Proof.

We want to compute $\zeta_{a,b}(\{4\}^n)$ from the generating function

$$\sum_{n\geq 0}\zeta_{a,b}(\{4\}^n)\,z^{4n} = \prod_{k\geq 1}\left(1+\frac{z^4}{z_{a,b;k}^4}\right)$$

$$= \prod_{k\geq 1}\left(1+\frac{(\sqrt{\imath}z)^2}{z_{a,b;k}^2}\right)\prod_{k\geq 1}\left(1+\frac{(\imath\sqrt{\imath}z)^2}{z_{a,b;k}^2}\right).$$

Since $\prod_{k\geq 1}\left(1+\frac{z^2}{z_{a,b;k}^2}\right) = {}_2F_3\left(\begin{matrix} a,b \\ a+b,\frac{a+b}{2},\frac{a+b+1}{2}\end{matrix};-\frac{z^2}{4}\right)$, we deduce

$$\sum_{n\geq 0}\zeta_{a,b}(\{4\}^n)\,z^{4n}$$

$$= {}_2F_3\left(\begin{matrix} a,b \\ a+b,\frac{a+b}{2},\frac{a+b+1}{2}\end{matrix};-\imath\frac{z^2}{4}\right){}_2F_3\left(\begin{matrix} a,b \\ a+b,\frac{a+b}{2},\frac{a+b+1}{2}\end{matrix};\imath\frac{z^2}{4}\right).$$

# Hypergeometric Zeta

Proof.

We have

$$
{}_2F_3\left(\begin{matrix} a_1, a_2 \\ b_1, b_2, b_3 \end{matrix}; cz\right) {}_2F_3\left(\begin{matrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2, \beta_3 \end{matrix}; dz\right) = \sum_{k\geq 0} \frac{z^k}{k!} d^k \frac{(\alpha_1)_k (\alpha_2)_k}{(\beta_1)_k (\beta_2)_k (\beta_3)_k}
$$

$$
\times {}_6F_5\left(\begin{matrix} -k, 1-k-\beta_1, 1-k-\beta_2, 1-k-\beta_3, a_1, a_2 \\ 1-k-\alpha_1, 1-k-\alpha_2, b_1, b_2, b_3 \end{matrix}; \frac{c}{d}\right).
$$

# Hypergeometric Zeta

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$$
\times\, {}_6F_5\left(\begin{matrix} -k, 1-k-\beta_1, 1-k-\beta_2, 1-k-\beta_3, a_1, a_2 \\ 1-k-\alpha_1, 1-k-\alpha_2, b_1, b_2, b_3 \end{matrix}; \frac{c}{d}\right).
$$

Therefore our desired hypergeometric product is

$$
\sum_{n\geq 0} \frac{z^{2n}}{n!} \left(\frac{\imath}{4}\right)^n \frac{(a)_n (b)_n}{(a+b)_n \left(\frac{a+b}{2}\right)_n \left(\frac{a+b+1}{2}\right)_n}
$$

$$
\times {}_6F_5\left(\begin{matrix} -n, 1-n-a-b, 1-n-\frac{a+b}{2}, 1-n-\frac{a+b+1}{2}, a, b \\ 1-n-a, 1-n-b, a+b, \frac{a+b}{2}, \frac{a+b+1}{2} \end{matrix}; -1\right)
$$

□

# Hypergeometric Zeta *

We want to compute now $\zeta^*_{a,b}(\{2\}^n)$ with generating function

$$\sum_{n\geq 0} \zeta^*_{a,b}(\{2\}^n) z^{2n} = \prod_{k\geq 1}\left(1-\frac{z^2}{z^2_{a,b;k}}\right)^{-1}$$

$$= \frac{1}{{}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; z\right) {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; -z\right)}.$$

# Hypergeometric Zeta *

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$$= \frac{1}{{}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; z\right) {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; -z\right)}.$$

Let us introduce the **hypergeometric Bernoulli numbers** $B_n^{(a,b)}$, defined through their generating function

$$\sum_{n\geq 0} \frac{B_n^{(a,b)}}{n!} z^n = \frac{1}{{}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}; z\right)}.$$

# Hypergeometric Zeta

We have

$$\prod_{k\geq 1}\left(1-\frac{z^2}{z_{a,b;k}^2}\right)^{-1} = \sum_{k,l\geq 0}\frac{B_k^{(a,b)}B_l^{(a,b)}}{k!l!}(-1)^k z^{k+l}$$

$$= \sum_{n\geq 0}\frac{z^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k B_k^{(a,b)}B_{n-k}^{(a,b)}.$$

so that

$$\zeta_{a,b}^{*}\left(\{2\}^n\right) = \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k B_k^{(a,b)}B_{2n-k}^{(a,b)}.$$

For example,

$$\zeta_{a,b}^{*}\left(\{2\}\right) = -\frac{ab}{(a+b)^2(1+a+b)} = \zeta_{a,b}\left(\{2\}\right)$$

and

$$\zeta_{a,b}^{*}\left(\{2\}^2\right) = \frac{ab\left(-a^2-a^3-b^2-b^3+ab\,(a+b+5)\,(a+b+2)\right)}{2(a+b)^4(1+a+b)^2(2+a+b)(3+a+b)}.$$

# Hypergeometric Zeta

## Theorem

*The hypergeometric Bernoulli numbers satisfy the linear recurrence*

$$\sum_{k=0}^{n} \binom{a+b+n-1}{k} \binom{a-1+n-k}{n-k} B_k^{(a,b)} = (a+b)_n \, \delta_n,$$

*with initial condition* $B_0^{(a,b)} = 1$.

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*with initial condition* $B_0^{(a,b)} = 1$.

This recurrence is a consequence of the identity

$$H(t)\, E(-t) = 1$$

and allows us to explicitly compute $B_n^{(a,b)}$ in terms of the lower-order numbers $\left\{ B_k^{(a,b)} \right\}_{k<n}$, and in turn to deduce the values of $\zeta_{a,b}^{*}\left(\{2p\}^{n}\right)$.

# Bessel Zeta

Consider

$$j_\nu(z) := 2^\nu \Gamma(\nu+1) \frac{J_\nu(z)}{z^\nu},$$

with the Weierstrass factorization

$$j_\nu(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,k}^2}\right), \ \forall z \in \mathbb{C}.$$

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Frappier developed an extensive theory centered around these functions. This led him to define a family of Bernoulli polynomials $B_{n,\nu}(x)$ called "$\alpha$-Bernoulli polynomials" with the generating function

$$\frac{e^{(x-\frac{1}{2})z}}{j_\nu(\frac{iz}{2})} = \sum_{n=0}^{\infty}\frac{B_{n,\nu}(x)}{n!}z^n.$$

# Hypergeometric Zeta

The case $\nu = \frac{1}{2}$ recovers the Riemann zeta function since

$$j_{\frac{1}{2}}(z) = \frac{\sin z}{z} = \prod_{k \geq 1} \left(1 - \frac{z^2}{k^2 \pi^2}\right),$$

$$z_{\frac{1}{2},k} = k\pi, \;\; \zeta_{B,\frac{1}{2}}(s) = \frac{1}{\pi^s}\zeta(s), \;\; B_{n,\frac{1}{2}}(z) = B_n(z).$$

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$$z_{\frac{1}{2},k} = k\pi,\ \zeta_{B,\frac{1}{2}}(s) = \frac{1}{\pi^s}\zeta(s),\ B_{n,\frac{1}{2}}(z) = B_n(z).$$

The case $\nu = -\frac{1}{2}$ corresponds to

$$j_{-\frac{1}{2}}(z) = \cos z = \prod_{k\geq 1}\left(1 - \frac{z^2}{\left(k+\frac{1}{2}\right)^2\pi^2}\right),$$

$$z_{-\frac{1}{2},k} = \left(k+\frac{1}{2}\right)\pi,\ \zeta_{B,-\frac{1}{2}}(s) = \frac{2^s-1}{\pi^s}\zeta(s)$$

and $B_{n,-\frac{1}{2}}(z)$ coincides with the Euler polynomial $E_n(z)$

$$\sum_{n\geq 0}\frac{E_n(z)}{n!}x^n = \frac{2e^{xz}}{e^x+1}.$$

# Hypergeometric Zeta

### Theorem

*The following evaluations hold for the zeta function built out of the Bessel zeros:*

$$\zeta_{B,\nu}(\{2\}^n) = \frac{1}{2^{2n} n!(\nu+1)_n},$$

$$\zeta^*_{B,\nu}(\{2\}^n) = \frac{B_{2n,\nu}(\frac{1}{2})}{(2n)!} 2^{2n}(-1)^n,$$

$$S_{B,\nu}(2n,k) = \sum_{r=k}^{n} (-1)^{n-k} \binom{r}{k} \frac{2^{2n-4r}\Gamma(\nu+1) B_{2n-2r,\nu}\left(\frac{1}{2}\right)}{r!(2n-2r)!\Gamma(\nu+r+1)},$$

$$S^*_{B,\nu}(2n,k) = (-1)^n \sum_{r=k}^{n} \binom{r}{k} \frac{\Gamma(\nu+1) B_{2r,\nu}\left(\frac{1}{2}\right)}{2^{2n-4r}(n-r)!(2r)!\Gamma(\nu+n-r+1)}.$$

# Hypergeometric Zeta

## Theorem

*The following evaluation holds:*

$$\zeta_{B,\nu}(\{4\}^n) = \frac{1}{2^{4n} n! \, (\nu+1)_{2n} \, (\nu+1)_n}.$$

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$$\zeta_{B,\nu}(\{4\}^n) = \frac{1}{2^{4n} n! \, (\nu+1)_{2n} \, (\nu+1)_n}.$$

## Proof.

Dissect the generating product as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \zeta_{B,v}(\{4\}^n) t^{2n} &= \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{z_{\nu,k}^4}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \zeta_{B,v}(\{2\}^n) t^n \sum_{n=0}^{\infty} \zeta_{B,v}(\{2\}^n) t^n.\end{aligned}$$

□

# Hypergeometric Zeta

## Proof.

Comparing coefficients of $t^n$ yields

$$\begin{aligned}\zeta_{B,\nu}(\{4\}^n) &= (-1)^n \sum_{l=0}^{2n} (-1)^l \zeta_{B,\nu}(\{2\}^l)\zeta_{B,\nu}(\{2\}^{2n-l})) \\ &= (-1)^n \sum_{l=0}^{2n} (-1)^l \frac{\Gamma^2(\nu+1)}{2^{4n}(2n-l)!l!\Gamma(\nu+l+1)\Gamma(\nu+2n-l+1)}.\end{aligned}$$

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$$=(-1)^n \sum_{l=0}^{2n} (-1)^l \frac{\Gamma^2(\nu+1)}{2^{4n}(2n-l)!l!\Gamma(\nu+l+1)\Gamma(\nu+2n-l+1)}.$$

This sum is now identified as a Gauss hypergeometric function

$$\zeta_{B,\nu}(\{4\}^n) = \frac{(-1)^n\Gamma(\nu+1)}{2^{4n}(2n)!\Gamma(2n+\nu+1)}\, {}_2F_1\left(\begin{matrix}-2n, -2n-\nu \\ \nu+1\end{matrix}; -1\right),$$

which can be evaluated using Kummer's identity as

$${}_2F_1\left(\begin{matrix}-2n, -2n-\nu \\ \nu+1\end{matrix}; -1\right) = (-1)^n \frac{2\Gamma(\nu+1)\Gamma(2n)}{\Gamma(n+\nu+1)\Gamma(n)}.$$

# Krein's expansion and an extension of Euler's identity

### Theorem

*Define the alternate Bessel zeta function* $\tilde{\zeta}_\nu$ *as*

$$\tilde{\zeta}_\nu\left(r\right) = \sum_{k\geq 1} \frac{1}{j_{\nu+1}\left(z_{\nu,k}\right) z_{\nu,k}^{r+2}}.$$

# Krein's expansion and an extension of Euler's identity

**Theorem**

*Define the alternate Bessel zeta function $\tilde{\zeta}_\nu$ as*

$$\tilde{\zeta}_\nu(r) = \sum_{k \geq 1} \frac{1}{j_{\nu+1}(z_{\nu,k})\, z_{\nu,k}^{r+2}}.$$

*Then for $n \geq \left[\frac{\nu}{2} + \frac{1}{4}\right] + 1$, the Bessel-Bernoulli polynomials can be expressed as*

$$\frac{(-1)^n 2^{2n}}{2n!} B_{2n,\nu}\left(\frac{1}{2}\right) = 4(\nu+1)\,\tilde{\zeta}_\nu(2n).$$

# Krein's expansion and an extension of Euler's identity

### Theorem

*Define the alternate Bessel zeta function* $\tilde{\zeta}_{\nu}$ *as*

$$\tilde{\zeta}_{\nu}\left(r\right) = \sum_{k\geq 1} \frac{1}{j_{\nu+1}\left(z_{\nu,k}\right) z_{\nu,k}^{r+2}}.$$

*Then for* $n \geq \left[\frac{\nu}{2}+\frac{1}{4}\right]+1$, *the Bessel-Bernoulli polynomials can be expressed as*

$$\frac{\left(-1\right)^{n} 2^{2n}}{2n!} B_{2n,\nu}\left(\frac{1}{2}\right) = 4\left(\nu+1\right)\tilde{\zeta}_{\nu}\left(2n\right).$$

*The case* $\nu = \frac{1}{2}$ *recovers Euler's identity*

$$\frac{\left(2\pi\right)^{2n}}{2\left(2n\right)!}\left(-1\right)^{n-1} B_{2n} = \zeta\left(2n\right).$$

# Krein's expansion and an extension of Euler's identity

Howard proved

$$\zeta_{r+\frac{1}{2}}(2n-2r+2) = (-1)^n \, 2^{2n-2r+1} \frac{(2r+1)!}{r!} \sum_{j=0}^{r-1} \frac{(2r-j-2)!}{j!\,(r-j-1)!} \frac{B_{2n-j+1,r}}{(2n-j+1)!}$$

generalized by K. Dilcher to the confluent case

$$\zeta_{a,b}(n-a-b+1) = (-1)^{b+1} \frac{(a+b+1)!}{n!\,(a+b)} \times \left[ \sum_{j=0}^{a-1} \binom{n}{j} \binom{a+b-j-2}{b-1} B_{n-j}^{(a,b)} - (-1)^n \sum_{j=0}^{b-1} \binom{n}{j} \binom{a+b-j-2}{a-1} B_{n-j}^{(a,b)} \right]$$

# Hypergeometric Zeta

## Proof.

Define

$$p := \left[\frac{\nu}{2} + \frac{1}{4}\right] + 1,$$

We have Krein's expansion

$$\frac{z^{\nu}}{J_{\nu}(z)} = P_p(z) - 2z^{2p} \sum_{k\geq 1} \frac{1}{J_{\nu+1}(z_{\nu,k})\, z_{\nu,k}^{2p-\nu-1} \left(z^2 - z_{\nu.k}^2\right)},$$

where $P_p(z)$ is the polynomial of degree $2p-2$ defined as the truncated Taylor expansion at 0 of $\frac{z^{\nu}}{J_{\nu}(z)}$:

$$P_p(z) = \sum_{m=0}^{p-1} \frac{d^{2m}}{dz^{2m}} \left(\frac{z^{\nu}}{J_{\nu}(z)}\right)_{|z=0} \frac{z^{2m}}{2m!}.$$

□

# Hypergeometric Zeta

## Proof.

Now take $|z| < |z_{\nu,1}|$, the smallest zero of $J_\nu$, so that

$$\frac{1}{z_{\nu,k}^{2p-\nu-1}\left(z^2 - z_{\nu.k}^2\right)} = -\frac{1}{z_{\nu,k}^{2p-\nu+1}\left(1 - \frac{z^2}{z_{\nu.k}^2}\right)} = -\sum_{q\geq 0} \frac{z^{2q}}{z_{\nu,k}^{2q+2p-\nu+1}}$$

and the infinite sum in Krein's expansion is

$$\sum_{k\geq 1} \frac{-2z^{2p-\nu}}{J_{\nu+1}\left(z_{\nu,k}\right) z_{\nu,k}^{2p-\nu-1}\left(z^2 - z_{\nu.k}^2\right)} = 2\sum_{q\geq 0} z^{2p+2q-\nu} \sum_{k\geq 1} \frac{1}{J_{\nu+1}\left(z_{\nu,k}\right) z_{\nu,k}^{2q+2p-\nu+1}}$$

Using the definition of the alternate Bessel zeta function, we obtain

$$\frac{1}{j_\nu(z)} = \frac{P_p(z)}{2^\nu \Gamma(\nu+1)} + 4(\nu+1)\sum_{q\geq 0} z^{2p+2q} \tilde{\zeta}_\nu(2q+2p).$$

□

# Hypergeometric Zeta

Remark : Since

$$\zeta^*_{B,\nu}(\{2\}^n) = \frac{B_{2n,\nu}(\frac{1}{2})}{(2n)!}2^{2n}(-1)^n,$$

we also have

$$\zeta^*_{B,\nu}(\{2\}^n) = 4\,(\nu+1)\,\tilde{\zeta}_\nu\,(2n)\,,$$

or

$$\sum_{k_1\geq k_2\geq\cdots\geq k_n\geq 1} \frac{1}{z^2_{\nu,k_1}\cdots z^2_{\nu,k_n}} = 4\,(\nu+1)\sum_{k\geq 1}\frac{1}{j_{\nu+1}\left(z_{\nu,k}\right)z^{2n+2}_{\nu,k}}.$$

# A Bessel-Gessel-Viennot identity

In [19], Hoffman uses generating functions to express the average $S(2n, k)$ in two different ways (in the case $\nu = \frac{1}{2}$). This yields, for $k \le n$,

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = \frac{(-1)^{n-k}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2^{2i} B_{2i}$$

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This a variation of the Gessel-Viennot identity

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k}\binom{2n+1}{2i+1} B_{2n-2i} = \frac{2n+1}{2}\binom{2k-2n}{k}, \;\; k > n,$$

that is valid on the complementary range $k > n$.

# A Bessel-Gessel-Viennot identity

**Theorem**

*The average $S_{B,\nu}(2n,k)$ can be expressed as, for $k \leq n$,*

$$S_{B,\nu}(2n,k) = \sum_{r=k}^{n} (-1)^{n-k} \binom{r}{k} \frac{2^{2n-4r}\Gamma(\nu+1)}{r!(2n-2r)!\Gamma(\nu+r+1)} B_{2n-2r,\nu}\left(\frac{1}{2}\right),$$

*or alternatively as, still for $k \leq n$,*

$$S_{B,\nu}(2n,k) = \frac{1}{k!} \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^j}{2^{2j}} \binom{k-1-j}{j} \frac{\Gamma(\nu+k-j)}{\Gamma(\nu+1+j)} \zeta_\nu(2n-2j).$$

The case $\nu = \frac{1}{2}$ recovers Hoffman's identity.

# A Bessel-Gessel-Viennot identity

The proof of this result uses generating functions. It gives us as a by-product another identity valid for $\left\lfloor \frac{k+1}{2} \right\rfloor < n \leq k-1$,

$$\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^j}{2^{2j}} \binom{k-1-j}{j} \frac{\Gamma(\nu+k-j)}{\Gamma(\nu+1+j)} \zeta_\nu(2n-2j) = 0.$$

# A Bessel-Gessel-Viennot identity

Consider the Lommel polynomials, that enter in the decomposition

$$J_{\nu+1}(z) = \frac{2\nu}{z} J_{\nu}(z) - J_{\nu-1}(z)$$

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$$\begin{aligned} J_{\nu+2}(z) &= \frac{2(\nu+1)}{z}\frac{2\nu}{z} J_\nu(z) - \frac{2(\nu+1)}{z} J_{\nu-1}(z) - J_\nu(z) \\ &= \left[\frac{2(\nu+1)}{z}\frac{2\nu}{z} - 1\right] J_\nu(z) - \frac{2(\nu+1)}{z} J_{\nu-1}(z) \end{aligned}$$

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More generally

$$j_{\nu+m}(z) = \frac{R_{m-1,\nu+1}(z)}{\left(\frac{z}{2}\right)^{m-1}} (\nu+2)_{m-1} j_{\nu+1}(z) - \frac{R_{m-2,\nu+2}(z)}{\left(\frac{z}{2}\right)^{m}} (\nu+1)_m j_\nu(z).$$

# A Bessel-Gessel-Viennot identity

For example

$$R_{0,\nu}(z) = 1,\ R_{1,\nu} = \frac{2\nu}{z},\ R_{2,\nu} = \frac{4\nu(\nu+1)}{z^2} \ldots$$

and

$$R_{n,\nu}(z) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n-j}{j} \frac{\Gamma(\nu+n-j)}{\Gamma(\nu+j)} \left(\frac{z}{2}\right)^{2j-n}.$$

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The Lommel polynomials are orthogonal with respect to the discrete measure

$$\rho(z) = \sum_{n\geq 1} \frac{1}{z_{\nu-1,n}^2} \left[\delta\left(z - \frac{1}{z_{\nu-1,n}}\right) + \delta\left(z + \frac{1}{z_{\nu-1,n}}\right)\right]$$

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For example

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$$\int_{-\infty}^{+\infty} R_{r,\nu}\left(\frac{1}{x}\right) R_{s,\nu}\left(\frac{1}{x}\right) \rho(x)\, dx = \frac{\delta_{r,s}}{2(\nu+r)}.$$

# A Bessel-Gessel-Viennot identity

Choosing $s \in [0, r]$ and expressing $x^s$ as a linear combination of the polynomials $R_{r,\nu}$ yields

$$\sum_{q=1}^{+\infty} \frac{R_{r,\nu}\left(z_{\nu-1,q}\right)}{z_{\nu-1,q}^{s+2}} = \frac{\Gamma\left(\nu\right)}{2^{r+2}\Gamma\left(\nu+r+1\right)}\delta_{r,s},\ 0 \leq s \leq r,$$

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This is equivalent to

$$\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^j}{2^{2j}} \binom{k-1-j}{j} \frac{\Gamma\left(\nu+k-j\right)}{\Gamma\left(\nu+1+j\right)} \zeta_\nu\left(2n-2j\right) = 0,\ \left\lfloor \frac{k+1}{2} \right\rfloor < n \leq k-1.$$

# Airy Zeta function

The Airy function has Weierstrass factorization

$$\mathrm{Ai}\,(z) = \mathrm{Ai}\,(0)\, e^{\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} z} \prod_{n \geq 1} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}},$$

where all the zeros $\{a_n\}$ are real and negative. We frequently use the constants

$$\mathrm{Ai}\,(0) = \frac{1}{3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}, \quad \mathrm{Ai}'\,(0) = -\frac{1}{3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}\right)}.$$

## Theorem

*The Airy MZV is equal to*

$$\zeta_{\mathrm{Ai}}\left(\{2\}^n\right) = \frac{1}{12^{\frac{n}{3}} n! \left(\frac{5}{6}\right)_{\frac{n}{3}}}.$$

# Airy Zeta function

The usual proof uses the series expansion

$$\mathrm{Ai}(x)\,\mathrm{Ai}(-x) = \frac{2}{\sqrt{\pi}} \sum_{n\geq 0} \frac{(-1)^n\, x^{2n}}{12^{\frac{2n+5}{6}}\, n!\, \Gamma\left(\frac{2n+5}{6}\right)},$$

deduced by Reid as a consequence of the integral representation

$$\mathrm{Ai}(x)\,\mathrm{Ai}(-x) = \frac{1}{\pi 2^{\frac{1}{3}}} \int_{-\infty}^{+\infty} \mathrm{Ai}\left(2^{-\frac{4}{3}} t^2\right) \cos(xt)\, dt.$$

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Then the Weierstrass factorization allows to deduce the generating function of $\zeta_{\mathrm{Ai}}(\{2\}^{\,n})$ as

$$\sum_{n=0}^{\infty} \zeta_{\mathrm{Ai}}(\{2\}^n)\, z^n = \prod_{n\geq 1} \left(1 + \frac{z}{a_n^2}\right) = \frac{\mathrm{Ai}(\imath\sqrt{z})\,\mathrm{Ai}(-\imath\sqrt{z})}{\mathrm{Ai}(0)^2}.$$

# Airy Zeta function

We observe that

$$\zeta_{B,-\frac{1}{3}}(\{4\}^n) = \left(\frac{3}{2}\right)^{4n} \zeta_{\mathrm{Ai}}\left(\{2\}^{3n}\right).$$

This is not a coincidence.

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This is not a coincidence.

## Theorem

*The Airy MZV and the Bessel MZV are related by*

$$\zeta_{\mathrm{Ai}}\left(\{2\}^{3n}\right) = \left(\frac{2}{3}\right)^{4n} \zeta_{B,-\frac{1}{3}}\left(\{4\}^n\right)$$

*and*

$$\frac{\zeta_{\mathrm{Ai}}\left(\{2\}^{3n+1}\right)}{\zeta_{\mathrm{Ai}}\left(\{2\}\right)} = \left(\frac{2}{3}\right)^{4n} \zeta_{B,\frac{1}{3}}\left(\{4\}^n\right).$$

# Airy Zeta function

## Proof.

The Airy function is an entire function: with $\xi = \frac{2}{3} z^{\frac{3}{2}}$

$$\mathrm{Ai}\left(z\right) = \sum_{n \geq 0} a_n z^n = \frac{\sqrt{z}}{3} \left( I_{-\frac{1}{3}}\left(\xi\right) - I_{\frac{1}{3}}\left(\xi\right) \right)$$

such that $a_{3n+2} = 0$. The coefficients $a_{3n}$ are provided by the term $\frac{\sqrt{z}}{3} I_{-\frac{1}{3}}\left(\xi\right)$, while the coefficients $a_{3n+1}$ arise from $\frac{\sqrt{z}}{3} I_{\frac{1}{3}}\left(\xi\right)$.
Similarly, the two terms in the expansion

$$\mathrm{Ai}\left(-z\right) = \frac{\sqrt{z}}{3} \left( J_{-\frac{1}{3}}\left(\xi\right) - J_{\frac{1}{3}}\left(\xi\right) \right)$$

provide the coefficients $a_{3n}$ and $a_{3n+1}$ respectively.
We deduce

$$\sum_{n \geq 0} \left(-1\right)^n \zeta_{\mathrm{Ai}}\left( \{2\}^{3n} \right) \left( \frac{3z}{2} \right)^{4n} = j_{-\frac{1}{3}}\left(\imath z\right) j_{-\frac{1}{3}}\left(z\right),$$

# Airy Zeta function

## Theorem

*The values of* $\zeta_{\mathrm{Ai}}(\{4\}^n)$ *can be computed as the convolution*

$$\zeta_{\mathrm{Ai}}(\{4\}^n) = \sum_{k=0}^{2n} \frac{1}{12^k 12^{2n-k}} \frac{(-1)^k}{k!(2n-k)!} \frac{1}{\left(\frac{5}{6}\right)_{\frac{k}{3}} \left(\frac{5}{6}\right)_{\frac{2n-k}{3}}}$$

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*and are equal to*

$$\begin{aligned}\zeta_{\mathrm{Ai}}\left(\{4\}^n\right)=&\frac{1}{12^{2n}}\Gamma^2\left(\frac{5}{6}\right)\frac{{}_4F_3\left(\begin{matrix}\frac{2}{3}-\frac{2n}{3},\frac{5}{6}-\frac{2n}{3},1-\frac{2n}{3},\frac{4}{3}-\frac{2n}{3}\\ \frac{4}{3},\frac{3}{2},\frac{5}{3}\end{matrix};-1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2n}{3}+\frac{1}{6}\right)\Gamma(2n-1)}\\&-\frac{6}{12^{2n}}\Gamma^2\left(\frac{5}{6}\right)\frac{{}_4F_3\left(\begin{matrix}\frac{1}{3}-\frac{2n}{3},\frac{1}{2}-\frac{2n}{3},\frac{2}{3}-\frac{2n}{3},1-\frac{2n}{3}\\ \frac{2}{3},\frac{7}{6},\frac{4}{3}\end{matrix};-1\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2n}{3}+\frac{1}{2}\right)\Gamma(2n)}\\&+\frac{1}{12^{2n}}\Gamma\left(\frac{5}{6}\right)\frac{{}_4F_3\left(\begin{matrix}\frac{1}{6}-\frac{2n}{3},\frac{1}{3}-\frac{2n}{3},\frac{2}{3}-\frac{2n}{3},-\frac{2n}{3}\\ \frac{1}{3},\frac{2}{3},\frac{5}{6}\end{matrix};-1\right)}{\Gamma\left(\frac{2n}{3}+\frac{5}{6}\right)\Gamma(2n+1)}.\end{aligned}$$

# Airy Bernoulli numbers

Define the Airy Bernoulli numbers $\mathcal{B}_n$ by the generating function

$$\sum_{n\geq 0} \frac{\mathcal{B}_n}{n!} z^n = \frac{\mathrm{Ai}\,(0)}{\mathrm{Ai}\,(z)}.$$

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For example,

$$\mathcal{B}_0 = 1,\ \mathcal{B}_1 = 3^{\frac{1}{3}} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} = -\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)},\ \frac{\mathcal{B}_2}{2!} = \mathcal{B}_1^2 = 3^{\frac{2}{3}} \frac{\Gamma^2\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)},$$

$$\frac{\mathcal{B}_3}{3!} = -\frac{1}{6} + 3\frac{\Gamma^3\left(\frac{2}{3}\right)}{\Gamma^3\left(\frac{1}{3}\right)},\ \frac{\mathcal{B}_4}{4!} = -\frac{3^{\frac{1}{3}}}{4} \frac{\Gamma\left(\frac{2}{3}\right)\left[\Gamma^3\left(\frac{2}{3}\right) - 12\Gamma^3\left(\frac{1}{3}\right)\right]}{\Gamma^4\left(\frac{1}{3}\right)}$$

# Airy Bernoulli numbers

## Theorem

*Define the sequence*

$$a_{3n} = \frac{(-1)^n (3n)!}{3^{2n} n! \left(\frac{2}{3}\right)_n}, \quad a_{3n+1} = \frac{(-1)^{n+1} \Gamma\left(\frac{2}{3}\right) (3n+1)!}{3^{2n+\frac{2}{3}} n! \Gamma\left(n+\frac{4}{3}\right)}, \quad a_{3n+2} = 0.$$

*The Airy Bernoulli numbers satisfy the linear recurrence*

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k a_{n-k} = n! \delta_n$$

*so that*

$$\mathcal{B}_n = \begin{cases} -\sum_{k=0}^{n-1} \binom{n}{k} \mathcal{B}_k a_{n-k}, & n > 0 \\ 1 & n = 0 \end{cases}.$$

# Airy Bernoulli numbers

Taking the derivative of

$$\sum_{n\geq 0}\frac{\mathcal{B}_n}{n!}z^n=\frac{\mathrm{Ai}\,(0)}{\mathrm{Ai}\,(z)}.$$

and writing

$$-\mathrm{Ai}\,(0)\,\frac{\mathrm{Ai}'\,(z)}{\mathrm{Ai}^2\,(z)}=-\frac{\mathrm{Ai}\,(0)}{\mathrm{Ai}\,(z)}\frac{\mathrm{Ai}'\,(z)}{\mathrm{Ai}\,(z)}.$$

yields

$$\frac{\mathcal{B}_{n+1}}{n!}=-\frac{\mathrm{Ai}'\,(0)}{\mathrm{Ai}\,(0)}\frac{\mathcal{B}_n}{n!}+\sum_{r=0}^{n-1}\frac{\mathcal{B}_r}{r!}\zeta_{\mathrm{Ai}}\,(n+1-r)\,,\ \ n\geq 1,$$

so that

$$\zeta_{\mathrm{Ai}}\,(n+1)=\frac{\mathrm{Ai}'\,(0)}{\mathrm{Ai}\,(0)}\frac{\mathcal{B}_n}{n!}+\frac{\mathcal{B}_{n+1}}{n!}-\sum_{r=1}^{n-1}\frac{\mathcal{B}_r}{r!}\zeta_{\mathrm{Ai}}\,(n+1-r)\,,\ \ n\geq 1.$$

# Airy Zeta function: there is more

M Belloni and R W Robinett, Constraints on Airy function zeros from quantum-mechanical sum rules , J. Phys. A: Math. Theor. 42 (2009) 075203.

Question: how to evaluate

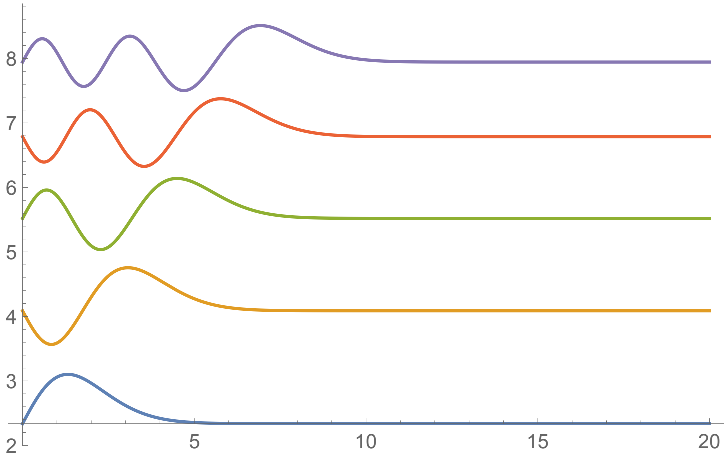
$$S_p(n) = \sum_{k \neq n} \frac{1}{(z_k - z_n)^p},$$

$$T_{p,q,r}(n) = \sum_{k \neq j \neq n} \frac{1}{(z_k - z_n)^p (z_k - z_j)^q (z_j - z_n)^r}$$

Consider the quantum bouncer

$$V(x) = \begin{cases} k.x & x \geq 0 \\ +\infty & x < 0 \end{cases}$$

# Airy Zeta function



# Airy Zeta function

The solution $\psi_n$ with energy $E_n$ of the Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n(x) + V(x)\psi_n(x) = E_n\psi_n(x)$$

is

$$\psi_n(x) = \frac{\mathrm{Ai}(x+z_n)}{|\,\mathrm{Ai}'(z_n)\,|}, \;\; E_n \propto -z_n.$$

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The solutions $\{\psi_n\}_{n\geq 0}$ form a complete set of orthonormal functions:

$$\int_0^{+\infty} \psi_n(z)\,\psi_k(z)\,dz = \delta_{n,k}.$$

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With quantum notations

$$\langle n|1|k\rangle = \delta_{n,k},$$

$$\langle n|f(z)|k\rangle = \int_0^{+\infty} \psi_n(z)\,\psi_k(z)\,f(z)\,dz.$$

# Airy Zeta function

Goodmanson gave a recurrence for these moments

$$\frac{p!}{(p-4)!}\left\langle n|x^{p-4}|k\right\rangle + 4p\,(p-1)\,\frac{z_n+z_k}{2}\left\langle n|x^{p-2}|k\right\rangle$$
$$-2p\,(2p-1)\left\langle n|x^{p-1}|k\right\rangle + (z_n-z_k)^2\left\langle n|x^{p}|k\right\rangle = 2\delta_{1,p}\,(-1)^{n-k+1}$$

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For example,

$$\langle n|x|n\rangle = \frac{2}{3}z_n,\ \left\langle n|x^2|n\right\rangle = \frac{8}{15}z_n^2,\ \left\langle n|x^3|n\right\rangle = \frac{16}{35}z_n^3 + \frac{3}{7}\ldots$$

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and

$$\langle n|x|k\rangle = \frac{2\,(-1)^{n-k+1}}{(z_k-z_n)^2},\ \left\langle n|x^2|k\right\rangle = \frac{24\,(-1)^{n-k+1}}{(z_k-z_n)^4},$$
$$\left\langle n|x^3|n\right\rangle = (-1)^{n-k+1}\left[\frac{720}{(z_k-z_n)^6} - \frac{48z_n}{(z_k-z_n)^4} - \frac{24}{(z_k-z_n)^3}\right]\ldots$$

# Airy Zeta function

Next use the Thomas-Reiche-Khun rule

$$\sum_{k \neq n} (E_n - E_k) \left| \langle n|x|k \rangle \right|^2 = \frac{\hbar^2}{2m}$$

and

$$E_k = -z_k$$

to deduce

$$\sum_{k \neq n} (z_k - z_n) \frac{4}{(z_k - z_n)^4} = 1$$

so that

$$S_3(n) = \sum_{k \neq n} \frac{1}{(z_k - z_n)^3} = \frac{1}{4}.$$

# Airy Zeta function

To obtain $S_4(n)$, write the closure relationship

$$\sum_{all\ \ k} \langle n|x|k\rangle \langle k|x|n\rangle = \langle n|x^2|n\rangle$$

so that

$$|\langle n|x|n\rangle|^2 + \sum_{k\neq n} \frac{4}{(z_k - z_n)^4} = \frac{8}{15} z_n^2 \Longrightarrow S_4(n) = \frac{z_n^2}{45}$$

# Airy Zeta function

Moreover, using

$$\sum_{all\ k} \sum_{all\ j} \langle n|x|k\rangle \langle k|x|j\rangle \langle j|x|n\rangle = \langle n|x^3|n\rangle$$

gives

$$T_{2,2,2}(n) = \sum_{k \neq j \neq n} \frac{1}{(z_k - z_n)^2 (z_k - z_j)^2 (z_j - z_n)^2} = \frac{2}{945} z_n^3 + \frac{5}{168}.$$

Compare to

$$\sum_{m,n \geq 1} \frac{1}{m^2 n^2 (m+n)^2} = \frac{\pi^6}{2835}.$$

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