

Two Sequences Related to Bernoulli and Euler Numbers

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May 30th, 2018

Two Sequences



$$1, 0, -\frac{1}{12}, 0, \frac{7}{240}, 0, -\frac{31}{1344}, 0$$

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$$1, 0, -1, 0, 5, 0, -61, 0$$

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Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \frac{2e^{zx}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}$$

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Bernoulli number $B_n = B_n(0)$ and Euler number $E_n = 2^n E_n(1/2)$.

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n	0	1	2	3	4
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
$B_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$
E_n	1	0	-1	0	5
$E_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x$	$x^3 - \frac{3}{2}x^2 + \frac{1}{4}$	$x^4 - 2x^3 + x$

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n	0	1	2	3	4	5	6	7	8
$B_n(\frac{1}{2})$	1	0	$-\frac{1}{12}$	0	$\frac{7}{240}$	0	$-\frac{31}{1344}$	0	$\frac{127}{3840}$

Guessing Formulas

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1, 2, 3,

Guessing Formulas

1, 2, 3, 5, 7

Guessing Formulas

1, 2, 3, 5, 7, 11,

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1, 2, 3, 5, 7, 11, 15, 22, ...

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Partition of numbers

$$p(4) = 5$$

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Partition of numbers

$$p(4) = 5$$

- ▶ $4 = 4$
- ▶ $4 = 3 + 1$
- ▶ $4 = 2 + 2$
- ▶ $4 = 2 + 1 + 1$
- ▶ $4 = 1 + 1 + 1 + 1$

Computer Algebra

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Theorem. Given polynomial $P(x)$ with $\deg P = d$, we have

$$\sum_{k=1}^n P(k) = Q(n),$$

for some polynomials $Q(x)$ with $\deg Q = d + 1$. 

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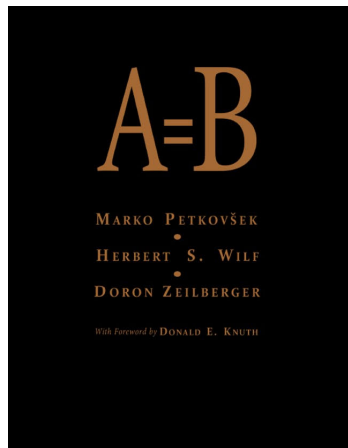
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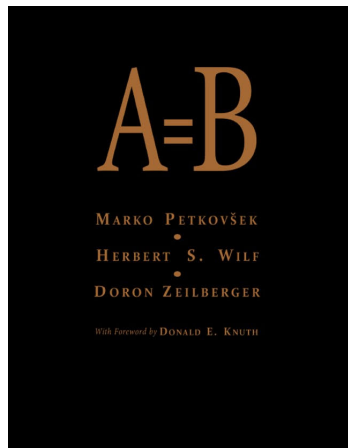
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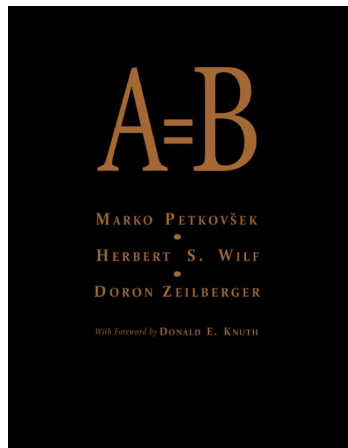
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$$\sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \binom{2k}{k} 4^{n-k} = \binom{2n}{n}$$

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Probabilistic interpretation of $B_n(x)$: Let L_B be a random variable with density function

$$p_B(t) = \frac{\pi}{2} \operatorname{sech}^2(\pi t) = \frac{\pi}{2} \left(\frac{1}{\cosh(\pi t)} \right)^2 \quad t \in \mathbb{R}$$

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In particular,

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- ▶ Moreover, P_n satisfies a three-term recurrence: for $n > 1$,

$$P_{n+1}(y) = (y + s_n)P_n(y) + t_n P_{n-1}(y).$$

Bernoulli

$$B_n = \mathbb{E} \left[\left(iL_B - \frac{1}{2} \right)^n \right] = \int_{\mathbb{R}} \left(it - \frac{1}{2} \right)^n p_B(t) dt.$$

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Theorem. (J. Touchard) For B_n , the orthogonal polynomials θ_n satisfy

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Proof. By induction on the degree of P .

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Proposition. Let $\phi_n = \sum \alpha_{n,k}y^{n-2k}$, then

$$\alpha_{n,k} = \sum_{\substack{i_1, \dots, i_k=1 \\ i_{j+1} - i_j > 1}}^n \omega_{i_1} \cdots \omega_{i_k},$$

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[Question2] Find the closed form for ϕ_n .

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$$\tau_0 = [\lambda_n, \dots, \lambda_0] \{ \text{last column of } (A_n^{-1}) \}, A_n = \begin{bmatrix} \alpha_{n,n} & 0 & \alpha_{n,n-1} & \cdots & \\ 0 & \alpha_{n-1,n-1} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Last Column: $1, 0, -\frac{1}{12}, 0, \frac{7}{240}, 0, -\frac{31}{1344}, 0, \frac{16473530237}{53003808000}, \dots$

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E_n	1	0	-1	0	5

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$$\begin{bmatrix} \varphi_4 \\ \varphi_3 \\ \varphi_2 \\ \varphi_1 \\ \varphi_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 14 & 0 & 9 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^4 \\ y^3 \\ y^2 \\ y \\ 1 \end{bmatrix}$$

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 P_n^{(\lambda)}(x; \phi) &:= \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \middle| 1 - e^{-2i\phi} \right) \\
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Example.

$$\varphi_n(y) = i^n n! P_n^{(\frac{1}{2})} \left(\frac{-iy}{2}; \frac{\pi}{2} \right)$$

End

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