The Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \operatorname{Re}(s) > 1$$

Special values:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n} (2\pi)^{2n}}{2 \cdot (2n)!} \qquad (\text{Euler})$$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi)$$

$$B_0 = 1, \; B_1 = -\frac{1}{2}, \; B_2 = \frac{1}{6}, \; B_4 = -\frac{1}{30}, \; B_6 = \frac{1}{42}$$

$$B_{2k+1} = 0, \quad (k \geq 1)$$

$\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$; simple pole at $s=1$

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \qquad (n \geq 0)$$

$$\zeta(0) = -\frac{1}{2}, \; \zeta(-1) = -\frac{1}{12}, \; \zeta(-2n) = 0, \; (n \geq 1)$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}} \qquad (\text{Euler product})$$

Gamma Function

Can we "interpolate" $n!$?

What is, e.g., "$\pi!$", or "$(-\sqrt{2})!$" ?

Define

$$\boxed{\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x} \, dx} \qquad (\text{Euler integral})$$

Converges when $\operatorname{Re}(z) > 0$.

Integration by parts:

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(1) = \int_0^{\infty} e^{-x} \, dx = 1$$

Induction: $\Gamma(n) = (n-1)!$

Analytic continuation to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Derivatives and special values of higher-order Tornheim zeta functions

Karl Dilcher

Dalhousie University

Number Theory Seminar
June 6, 2018

Joint work with

Hayley Tomkins

(University of Ottawa; formerly Dalhousie)

Partly based on earlier work with Jon Borwein (1951–2016)

1. Introduction

One of the best-known multiple zeta functions:

$$\mathcal{W}(r,s,t) := \sum_{m,n\geq 1} \frac{1}{m^r}\frac{1}{n^s}\frac{1}{(m+n)^t}.$$

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Witten (1991) studied a wider class; Zagier (1993) called them *Witten zeta functions*. Also often called *Mordell-Tornheim-Witten sums*.

$\mathcal{W}(r,s,t)$ can be analytically continued, separately in each of the three variables, to all of $\mathbb{C}^3$; poles at

$$r+s+t=2, \quad r+t=1-\ell, \quad s+t=1-\ell \quad (\ell \in \mathbb{N} \cup \{0\}).$$

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$$\omega_3(4) = \frac{19}{273648375}\pi^{12}.$$


OMEGA 3


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Columbus
Omega 3
Helps maintain a healthy
heart and supple joints
FREE RANGE EGGS
Columbus
Omega 3

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$$\omega_3(5) = -\frac{2}{9}\pi^4\zeta(11) - \frac{70}{3}\pi^2\zeta(13) + 252\,\zeta(15).$$

Also, $\omega_3(s)$ has simple poles at

$$s = \frac{2}{3} \quad \text{and} \quad s = \frac{1}{2} - k, \quad k = 0, 1, 2, \ldots$$

and no other singularities.
(Romik, 2015, who also determined the residues).

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$$\omega_3(0) = \frac{1}{3}.$$

Here, we'll give a different proof.

Dan Romik (UC Davis)

2. In the same paper, Romik showed:

$$\omega_3'(0) = \frac{1}{12}(1+\gamma) + \frac{3}{4}\log(2\pi) - 2\zeta'(-1) + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\zeta(\frac{3}{2}+it)\zeta(-\frac{3}{2}-it)}{(\frac{3}{2}+it)\cosh(\pi t)}dt$$
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Jon Borwein used high-precision evaluations and the integer relation algorithm PSLQ to conjecture the stunningly simple expression

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The main part of this talk concerns proving this and a generalization.

2. Some special functions

For each $s \in \mathbb{C}$, the *polylogarithm of order s* is defined by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} \qquad (|z| < 1).$$

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Special cases:

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Lemma

For $s \in \mathbb{C} \setminus \mathbb{N}$, and for $|\log z| < 2\pi$,

$$\mathrm{Li}_s(z) = \sum_{m=0}^{\infty} \zeta(s-m) \frac{\log^m z}{m!} + \Gamma(1-s)(-\log z)^{s-1}.$$

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$$\Gamma(t)\,\mathcal{W}(r,s,t) = \int_0^\infty x^{t-1}\mathrm{Li}_r(e^{-x})\mathrm{Li}_s(e^{-x})\mathrm{d}x.$$

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Proof: Use Euler's integral for $\Gamma(s)$ with an easy substitution:

$$\Gamma(s) = n^s \int_0^\infty e^{-nx}x^{s-1}\mathrm{d}x.$$

Replace s by t and n by $n+m$:

$$\frac{1}{(n+m)^t} = \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} e^{-(n+m)x} \mathrm{d}x \qquad (\mathrm{Re}(t) > 0).$$

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Plug into definition of $\mathcal{W}(r, s, t)$ and change order of summation and integration (legitimate):

$$\begin{aligned} \mathcal{W}(r,s,t) &= \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} \left(\sum_{n=1}^\infty \frac{e^{-nx}}{n^r} \right) \left(\sum_{m=1}^\infty \frac{e^{-mx}}{m^s} \right) \mathrm{d}x \\ &= \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} \mathrm{Li}_r(e^{-x}) \mathrm{Li}_s(e^{-x}) \mathrm{d}x. \end{aligned}$$

QED

3. Crandall's free parameter formula

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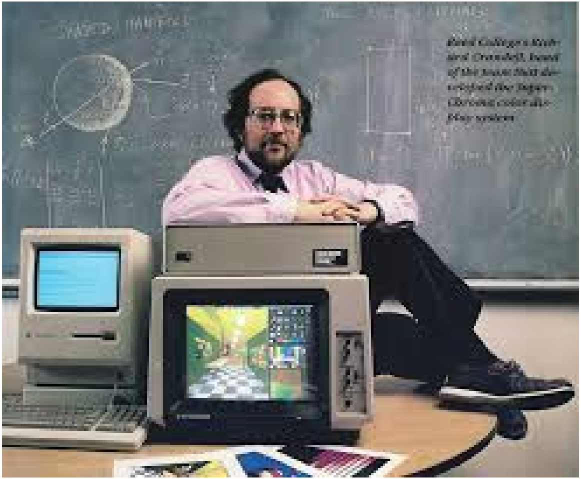

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We first need another special function:
The (upper) *incomplete Gamma function* is defined by

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Obviously, $\Gamma(a,0) = \Gamma(a)$.

Theorem (Crandall)

Let r, s, t be complex variables with $r \notin \mathbb{N}$ and $s \notin \mathbb{N}$. Then for any real $\theta > 0$ we have

$$\begin{aligned}
\Gamma(t)\mathcal{W}(r,s,t) = &\sum_{m,n\geq 1} \frac{\Gamma(t,(m+n)\theta)}{m^r n^s (m+n)^t} \\
&+ \sum_{u,v\geq 0} (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)\theta^{u+v+t}}{u!v!(u+v+t)} \\
&+ \Gamma(1-r)\sum_{q\geq 0}(-1)^q \frac{\zeta(s-q)\theta^{r+q+t-1}}{q!(r+q+t-1)} \\
&+ \Gamma(1-s)\sum_{q\geq 0}(-1)^q \frac{\zeta(r-q)\theta^{s+q+t-1}}{q!(s+q+t-1)} \\
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3. However, these singularities cancel, and a careful analysis gives a (more complicated) identity valid for *all* $r, s, t \in \mathbb{C}$.

4. This, and the above theorem, gives another analytic continuation to all of $\mathbb{C}^3$, with the exception of the known singularities.

Sketch of proof: Use defining integral of $\Gamma(a, z)$. A simple substitution gives

$$\int_\theta^\infty x^{t-1} e^{-(m+n)x} \mathrm{d}x = \frac{\Gamma(t, (m+n)\theta)}{(m+n)^t}.$$

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Use same argument as in the 2nd Lemma; break up integral:

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\Gamma(t)\mathcal{W}(r, s, t) &= \sum_{m,n\geq 1} \frac{1}{m^r n^s} \left(\int_0^\theta + \int_\theta^\infty \right) x^{t-1} e^{-(m+n)x} \mathrm{d}x \\
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Use the first Lemma, namely

$$\mathrm{Li}_s(z) = \sum_{m=0}^{\infty} \zeta(s-m) \frac{\log^m z}{m!} + \Gamma(1-s)(-\log z)^{s-1}.$$

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Expand and then integrate. QED

First application: Set $r = s = t$; then

$$\begin{aligned}\Gamma(s)\omega_3(s) &= \sum_{m,n\geq 1}\frac{\Gamma(s,(m+n)\theta)}{(mn(m+n))^s}\\ &\quad+\sum_{u,v\geq 0}(-1)^{u+v}\frac{\zeta(s-u)\zeta(s-v)\theta^{u+v+s}}{u!v!(u+v+s)}\\ &\quad+2\Gamma(1-s)\sum_{q\geq 0}(-1)^q\frac{\zeta(s-q)\theta^{2s+q-1}}{q!(2s+q-1)}\\ &\quad+\Gamma(1-s)^s\frac{\theta^{3s-2}}{3s-2}.\end{aligned}$$

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Together: $\omega_3(0) = 1/3$.

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and in particular,

$$\omega_3(0) = \omega_3(0;1) = \frac{1}{3}.$$

4. Derivative at the origin

Let's return to Crandall's identity; multiply both sides by s:

$$
\begin{aligned}
s\Gamma(s)\omega_3(s) = s \Bigg[& \sum_{m,n\geq 1} \frac{\Gamma(s,(m+n)\theta)}{(mn(m+n))^s} \\
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\end{aligned}$$

Now isolate the singularities in s in the large brackets on the RHS; bring them to the left:

$$
\begin{aligned}
s\Gamma(s)\omega_3(s) - \zeta(s)^2\theta^2 + \Gamma(1-s)\zeta(s-1)\theta^{2s} \\
= s\Bigg[\sum_{m,n\geq 1} \frac{\Gamma(s,(m+n)\theta)}{(mn(m+n))^s} \\
+2\Gamma(1-s)\zeta(s)\frac{\theta^{2s-1}}{2s-1} + \Gamma(1-s)^2\frac{\theta^{3s-2}}{3s-2} \\
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Derivative at $s=0$ of LHS becomes

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\omega_3'(0) - \tfrac{5}{12}\gamma - \tfrac{1}{2}\log 2\pi - \tfrac{5}{12}\log\theta + \zeta'(-1).
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$$\int_0^\infty \frac{t^{\alpha - 1}}{(e^t - 1)^2} dt = \Gamma(\alpha)(\zeta(\alpha - 1) - \zeta(\alpha)) \qquad (\mathrm{Re}(\alpha) > 2).$$

(An integral in Gradshteyn & Ryzhik).

Everything put together, we get (after some work)

$$\sum_{m,n\geq 1} \Gamma(0,(m+n)\theta) = \frac{1}{2}\log(2\pi) - \frac{5\gamma}{12} + \zeta'(-1) - \frac{1}{\theta} + \frac{1}{2\theta^2} - \frac{5}{12}\log\theta + O(\theta).$$

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As before, with small modifications we get more generally

$$\omega_3'(0;\tau) = \frac{\tau+1}{2}\log(2\pi) + \frac{(\tau-1)\tau}{\tau+1}\zeta'(-1).$$

(Recall: $\omega_3(s;\tau) := \mathcal{W}(s,s,\tau s)$.)

5. Extensions

1. A multi-dimensional analogue:

For $n \geq 2$ define

$$\mathcal{W}(r_1, \ldots, r_n, t) := \sum_{m_1, \ldots, m_n \geq 1} \frac{1}{m_1^{r_1} \ldots m_n^{r_n} (m_1 + \ldots m_n)^t}$$

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Hayley Tomkins (honours thesis, 2016) showed that

$$\omega_{n+1}(0) = \frac{(-1)^n}{n+1}$$

holds for $n \leq 7$, and conjectured that it is true for all n.

Meanwhile proved, using higher-order convolution identities for Bernoulli numbers and polynomials.
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A more complicated convolution identity shows that

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for all integers $n \geq 2$ and $k \geq 1$.

This is analogous to the zeta function identity

$$\zeta(-k) = 0$$

for $k = 2, 4, 6, \ldots$.

Also proved by Tomkins:

$$\begin{aligned}\omega_4'(0) &= -\log(2\pi) + \zeta'(-2)\\ &= -\log(2\pi) - \frac{\zeta(3)}{4\pi^2}.\end{aligned}$$

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How about $\omega_{n+1}'(0)$ for $n \geq 4$?

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Bailey & Borwein found experimentally:

$$\omega_5'(0) = \log(2\pi) - 2\zeta'(-2)$$
$$\omega_6'(0) = -\log(2\pi) + \tfrac{35}{12}\zeta'(-2) + \tfrac{1}{12}\zeta'(-4)$$
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$$\omega_{19}'(0) = \log(2\pi) - \tfrac{344499373}{33633600}\zeta'(-2) - \ldots - \tfrac{1}{1162377216000}\zeta'(-16).$$

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What are the coefficients in these expressions?

Theorem

For any $n \geq 2$ we have

$$\omega'_{n+1}(0) = (-1)^n \log(2\pi) + \tfrac{2}{(n-1)!} \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} s(n, 2j+1)\zeta'(-2j),$$

where $s(n,k)$ are the Stirling numbers of the first kind.

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2. Character analogues

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In analogy to $\omega_3(s)$, consider $\alpha_3(s) := \mathcal{A}(s,s,s)$.

Analogue to Crandall's formula is much simpler:

$$\Gamma(s)\alpha_3(s) = \sum_{m,n\geq 1} \frac{(-1)^m}{m^r}\frac{(-1)^n}{n^s}\frac{\Gamma(s,(m+n)\theta)}{(m+n)^s} + \sum_{u,v\geq 0}(-1)^{u+v}\frac{\eta(s-u)\eta(s-v)\theta^{u+v+s}}{u!v!(u+v+s)}.$$

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$$\alpha_3(0) = \eta(0)^2 = \frac{1}{4}.$$

Furthermore, using methods of before:

$$\begin{aligned} \alpha_3'(0) &= 2\eta'(0) - \eta'(-1) - \tfrac{1}{4}\gamma \\ &= \log(2\pi) - \tfrac{5}{3}\log 2 - \tfrac{1}{4}\gamma + 3\zeta'(-1). \end{aligned}$$

Character analogues in general:

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Many of the results in this talk were first obtained experimentally before they were proved.

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General Remark:

Many of the results in this talk were first obtained experimentally before they were proved.

Knowing what to expect provides a great deal of guidance, as well as certainty when it's done.

Thank you