A Linear Algebra Problem Related to Legendre Polynomials

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• Introduction: Statement of the Original Problem

Find $f(x)$ such that

$$g(\tfrac{1}{2}) = \int_0^1 f(x)g(x)dx$$

where $f(x)$ and $g(x)$ are polynomials of degree ≤ 2.

To solve let

$$f(x) = ax^2 + bx + c$$

and

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Then we just allow the coefficients of $\alpha, \beta, and \gamma$ to be equal and solve a system of equations.

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After finding this solution I wanted to have some fun and see how the answer changed if I made a small change to the original problem.

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Same thing again.

I continued for a few more degrees, and the pattern continued as well.

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To answer this we will state the question again, but in more general terms.

Generalizing the Problem

Find $f_n(x)$ such that

$$g(c) = \int_0^1 f_n(x)g(x)dx$$

where $g(x)$ and $f_n(x)$ are polynomials of degree $\leq n$.

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where $g(x)$ and $f_n(x)$ are polynomials of degree $\leq n$.

Now let us write our question in terms of the following proposition.

Proposition

Let $f_n(x)$ be as previously defined. Then when $c = \frac{1}{2}$ we have $f_{2m+1}(x) = f_{2m}(x)$ for $m \in \mathbb{N}$.

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Solving in the same manner leads to

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 \\ c \\ c^2 \\ \vdots \\ c^n \end{bmatrix}$$

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\vdots & \vdots & \vdots & \ddots & \vdots \\
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$$H\underline{a} = \underline{c}.$$

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The choice of H is because matrices of this form ($H_{i,j} = \frac{1}{i+j-1}$) are known as Hilbert matrices.

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$$(H_{i,j})^{-1} = (-1)^{i+j-1}(i+j-1)\binom{n+i}{i+j-1}\binom{n+j}{i+j-1}\binom{i+j-2}{i-1}^2.$$

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Before continuing we need a definition.

Definition

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These polynomials determine the coefficients of $f_n(x)$. That is,

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Let's do an example to clarify.

Example: If $n = 2$, then

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

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Using our formula,

$$H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

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So we have $n = 2$ and can see that $1 \leq i \leq 3$. Thus

$$\begin{aligned} h_{2,1}(c) &= 9 - 36c + 30c^2, \\ h_{2,2}(c) &= -36 + 192c - 180c^2, \\ h_{2,3}(c) &= 30 - 180c + 180c^2, \\ f_n(x) &= h_{2,1}(c) + h_{2,2}(c)x + h_{2,3}(c)x^2 \end{aligned}$$

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If n is odd, and $f_n(x) = f_{n-1}(x)$ when $c = \frac{1}{2}$, then the degree of $f_n(x)$ is $n - 1$. This means that $a_n = 0$ in $f_n(x)$.

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This would correspond to the polynomial formed by the bottom row of H^{-1} having a root at $c = \frac{1}{2}$. Using our definition, this can be written as $h_{n,n+1}(c)$ having a root at $c = \frac{1}{2}$.

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Using the formula for H^{-1} we can write $h_{n,n+1}(c)$ as

$$h_{n,n+1}(c) = \sum_{j=1}^{n+1} (-1)^{n+j+1}(n+j)\binom{2n+1}{n+j}\binom{n+j}{n+j}\binom{n+j-1}{n}^2 c^{j-1}$$

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All we need now is a definition to solve our problem.

Definition

The shifted Legendre polynomials, denoted $\tilde{P}_n(x)$, are given by

$$\tilde{P}_n(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k.$$

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Looking back at our expresion for $h_{n,n+1}(c)$,

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we can now see that $h_{n,n+1}(c)$ is just a multiple of $\tilde{P}_n(c)$.

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The shift is given by sending x to $2x - 1$. That is, if we denote the Legendre polynomials by $P_n(x)$, then $P_n(2x - 1) = \tilde{P}_n(x)$.

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The Legendre polynomials are known to have $x = 0$ as a root when their degree is odd. Therefore, the shifted Legendre polynomials must have a root at $x = \frac{1}{2}$ when their degree is odd.

So, $h_{n,n+1}(c) = a_n$ and these are just multiples of the shifted Legendre polynomials.

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The shifted Legendre polynomials have a root at $\frac{1}{2}$ when their degree is odd.

Therefore, if n is odd and $c = \frac{1}{2}$, then $a_n = 0$. This ends up forcing $f_n(x) = f_{n-1}(x)$, thus answering our question.

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If $h_{n,n+1}(c)$ is always just a multiple of a shifted Legendre polynomial, what do other rows correspond to?

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If $h_{n,n+1}(c)$ is always just a multiple of a shifted Legendre polynomial, what do other rows correspond to?

That is, how does $h_{n,1}(c)$ change as we change n? $h_{n,2}(c)$? etc.

• Another Approach to the Problem and the Other Rows of H^{-1}

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Theorem

(Riesz Represention Theorem) *If we have some finite dimensional vector space, V, and some linear functional ϕ on V, then there is a unique vector $u \in V$ such that*

$$\phi(v) = \langle v, u \rangle$$

for all $v \in V$.

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We can interpret our problem in terms of this theorem. The integral is an inner product, $g(x)$ corresponds to v, $f_n(x)$ corresponds to u, and evaluation at c is a linear functional.

This theorem has the consequence of allowing us to write

$$f_n(x) = \sum_{k=0}^{n} \frac{\tilde{P}_k(c)\tilde{P}_k(x)}{\int_0^1 \tilde{P}_k(x)^2 dx}$$

$$= \sum_{k=0}^{n} (2k+1)\tilde{P}_k(c)\tilde{P}_k(x)$$

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In this form $f_n(x)$ would be called the kernel of the shifted Legendre polynomials. Therefore what I am studying can be interpreted as looking at how the coefficients of this kernel change with n, and with c.

Moving back to $h_{n,i}(c)$, we can use the previous expression of $f_n(x)$ and the fact that

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Using this equation, I wanted to find a generating function for these polynomials.

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Now we need to make use of a relationship between the shifted Legendre polynomials.

$$(n+1)\tilde{P}_{n+1}(x) = (2n+1)(2x-1)\tilde{P}_n(x) - n\tilde{P}_{n-1}(x)$$

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If $i = 1$ we have

$$h_{n,1}(c) = \sum_{k=0}^{n} (-1)^k (2k+1) \tilde{P}_k(c).$$

Now we need to make use of a relationship between the shifted Legendre polynomials.

$$(n+1)\tilde{P}_{n+1}(x) = (2n+1)(2x-1)\tilde{P}_n(x) - n\tilde{P}_{n-1}(x)$$

Combining the two expressions, it can be shown that

$$h_{n,1}(c) = \frac{(-1)^n(n+1)}{2c}\left(\tilde{P}_n(c) + \tilde{P}_{n+1}(c)\right)$$

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Using this we can find a generating function for $h_{n,1}(c)$.

Rearranging, multiplying by x^{n+1}, and summing over n yields

$$\sum_{n=0}^{\infty}\frac{h_{n,1}(c)x^{n+1}}{n+1} = -\sum_{n=0}^{\infty}\frac{1}{2c}\left(\tilde{P}_n(c)(-x)^{n+1} + \tilde{P}_{n+1}(c)(-x)^{n+1}\right)$$

$$= -\frac{1}{2c}\left(-x\sum_{n=0}^{\infty}\tilde{P}_n(c)(-x)^n + \sum_{n=0}^{\infty}\tilde{P}_n(c)(-x)^n - \tilde{P}_0(c)\right)$$

$$= -\frac{1}{2c}\left((1-x)\sum_{n=0}^{\infty}\tilde{P}_n(c)x^n - 1\right)$$

$$= -\frac{1}{2c}\left(\frac{1-x}{\sqrt{1+2(2c-1)x+x^2}} - 1\right).$$

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$$\begin{aligned}\sum_{n=0}^{\infty}\frac{h_{n,1}(c)x^{n+1}}{n+1} &= -\sum_{n=0}^{\infty}\frac{1}{2c}\left(\tilde{P}_n(c)(-x)^{n+1}+\tilde{P}_{n+1}(c)(-x)^{n+1}\right)\\ &= -\frac{1}{2c}\left(-x\sum_{n=0}^{\infty}\tilde{P}_n(c)(-x)^n+\sum_{n=0}^{\infty}\tilde{P}_n(c)(-x)^n-\tilde{P}_0(c)\right)\\ &= -\frac{1}{2c}\left((1-x)\sum_{n=0}^{\infty}\tilde{P}_n(c)x^n-1\right)\\ &= -\frac{1}{2c}\left(\frac{1-x}{\sqrt{1+2(2c-1)x+x^2}}-1\right).\end{aligned}$$

Now we take the derivative with respect to x of both sides which gives us the generating function.

Let $\mathcal{H}_1(x)$ be the generating function for $h_{n,1}(c)$.

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Then we have from the previous slide

$$\mathcal{H}_1(x) = -\frac{1}{2c}\frac{d}{dx}\left(\frac{1-x}{\sqrt{1+2(2c-1)x+x^2}} - 1\right)$$
$$= \frac{1+x}{(1+2(2c-1)x+x^2)^{\frac{3}{2}}}$$

Now using our expression for $\mathcal{H}_1(x)$, along with

$$h_{n,i}(c) = \sum_{k=i-1}^{n} (-1)^{k+i-1} \binom{k}{i-1} \binom{k+i-1}{i-1} (2k+1) \tilde{P}_k(c)$$

we can find an expression for the generating function of any $h_{n,i}(c)$, denoted $\mathcal{H}_i(x)$.

Sparing the details of the calculation, as it is more complicated but similar to the derivation of $\mathcal{H}_1(x)$, the final final result is given by

$$\mathcal{H}_i(x) = \frac{(-x)^{i-1}}{(1-x)((i-1)!)^2} \frac{d^{2i-2}}{dx^{2i-2}} \left(\frac{x^{i-1}(1-x)(1+x)}{(1+2(2c-1)x+x^2)^{\frac{3}{2}}} \right)$$

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If we let $j = i - 1$ then this takes on a nicer form of

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So we have accomplished our goal of finding the generating function for the polynomials $h_{n,i}(c)$.

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The part I want to show however, is the polynomial in the numerator.

For $\mathcal{H}_1(x)$ the polynomial would just be 1.

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For $\mathcal{H}_2(x)$ the polynomial would be

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For $\mathcal{H}_2(x)$ the polynomial would be

$$\left((c-\tfrac{1}{2})x^2-(c^2-c-1)x+c-\tfrac{1}{2}\right).$$

And for $\mathcal{H}_3(x)$

$$\left(c^2-c+\tfrac{1}{6}\right)x^4+\left(-\tfrac{4}{3}c^3+2c^2+\tfrac{2}{3}c-\tfrac{2}{3}\right)x^3$$

$$-\tfrac{1}{3}(c^2-c+3)(c^2-c-1)x^2$$

$$+\left(-\tfrac{4}{3}c^3+2c^2+\tfrac{2}{3}c-\tfrac{2}{3}\right)x+\left(c^2-c+\tfrac{1}{6}\right).$$

Other Results

Now I will quickly state some other results and properties of these polynomials.

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First, the polynomials $h_{n,i}(c)$ have an interesting property with the inner product from which they came.

$$\int_0^1 h_{n,i}(c)c^n dc = \begin{cases} 1 & \text{if } i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

In other words, integrating $h_{n,i}(c)$ against another polynomial in c of degree at least n, will be equal to the coefficient of c^{i-1} of the polynomial.

Another representation for these polynomials is

$$h_{n,i}(c) = (-1)^i i \binom{n+i}{i} \binom{n+1}{i} {}_3F_2(-n, n+2, i; 1, i+1; c)$$

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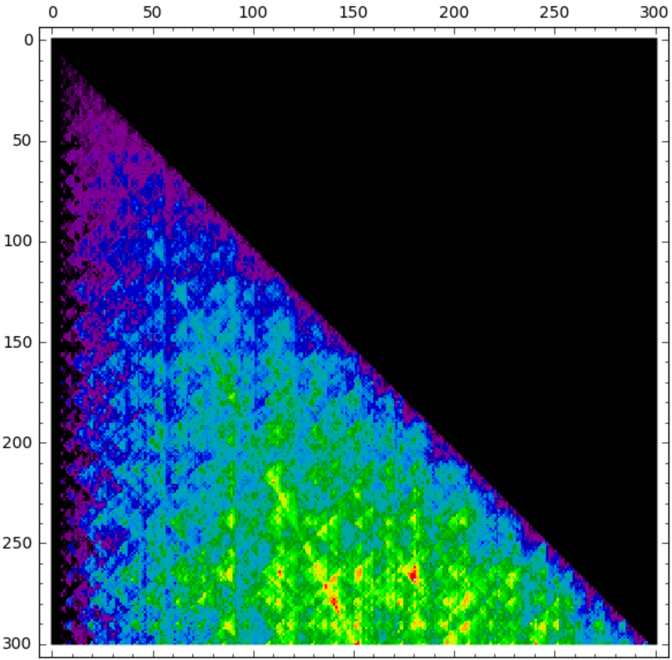
This just gives us another representation of the polynomials which can be investigated. Given this representation I was also able to prove the following identity

$$\int_0^1 {}_3F_2(-n, n+2, i; 1, i+1; c)\,{}_3F_2(-m, m+2, i; 1, i+1; c)dc$$
$$= \frac{i^2\Gamma(n+i+1)\Gamma(m+2-i)}{(2i-1)(n+1)(m+1)\Gamma(n+2-i)\Gamma(m+i+1)}$$

Next, I would like to show you some images that arise from the hypergeometric representation of the polynomials.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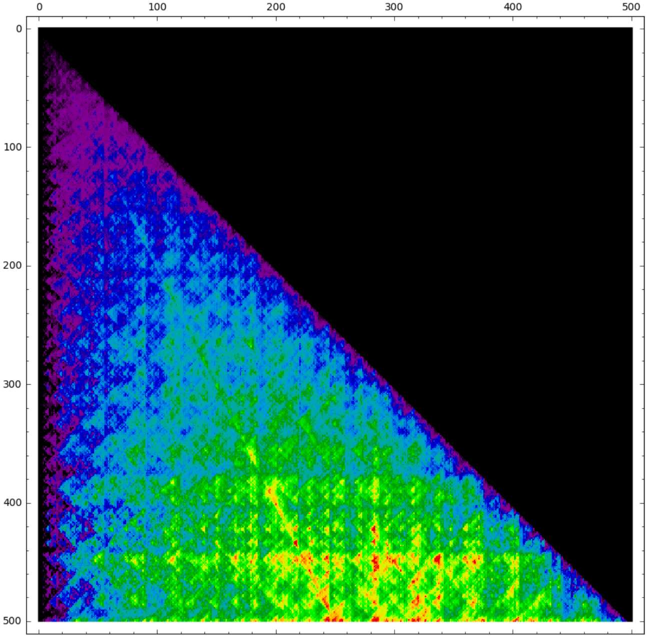
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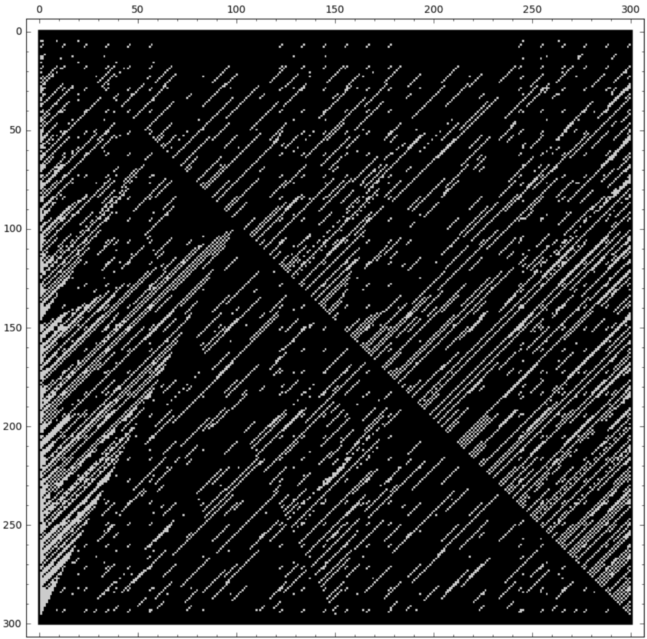
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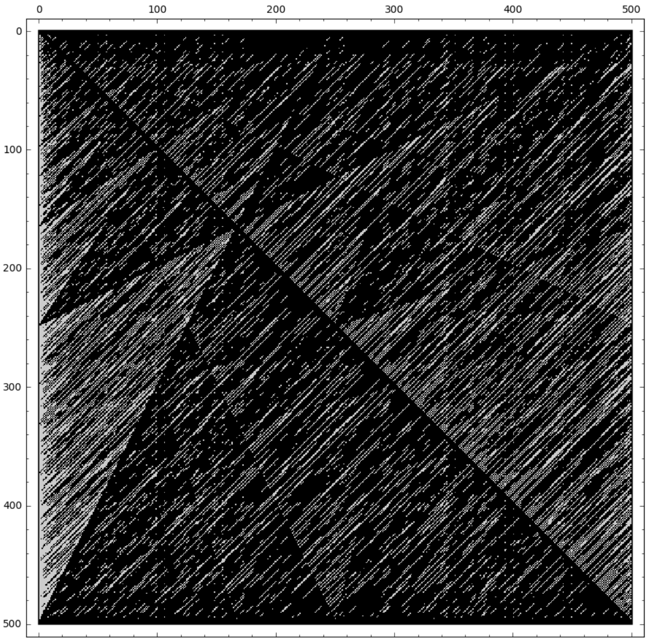
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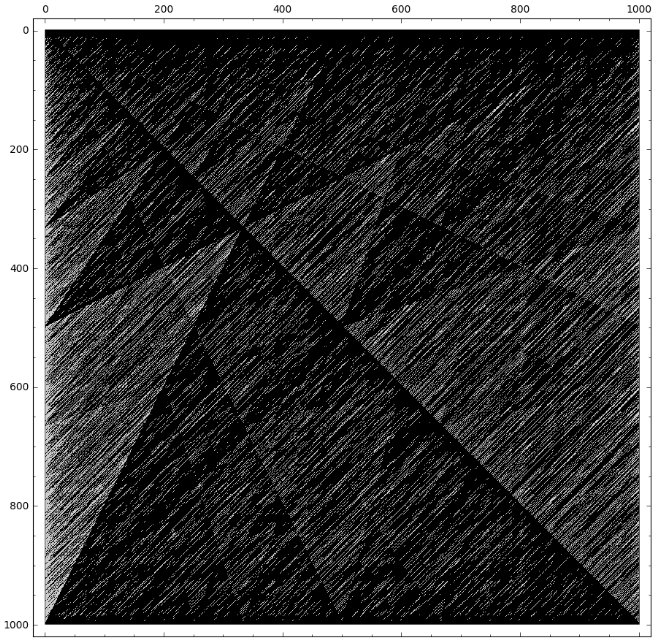
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Thanks for listening.

Special thanks to Dalhousie University
and to my advisor, Karl Dilcher