p-adic valuations of certain colored partition functions

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Short plan of the presentation

- The general question

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- The Prouhet-Thue-Morse sequence and the binary partition function

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- The Prouhet-Thue-Morse sequence and the binary partition function
- A general result
- 2-adic valuations for all powers
- Some results for p-ary colored partitions

The general question

In the sequel we will use the following notation:

- $\mathbb{N}$ denote the set of non-negative integers,
- $\mathbb{N}_{+}$ - the set of positive integers,
- $\mathbb{P}$ - the set of prime numbers,
- $\mathbb{N}_{\geq k}$ - the set $\{n \in \mathbb{N} : \ n \geq k\}$.

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If $p \in \mathbb{P}$ and $n \in \mathbb{Z}$ we define the p-adic valuation of n as:

$$\nu_p(n) := \max\{k \in \mathbb{N} : \ p^k \mid n\}.$$

We also adopt the standard convention that $\nu_p(0) = +\infty$.

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From the definition we easily deduce that for each $n_1, n_2 \in \mathbb{Z}$ the following properties hold:

$$\nu_p(n_1 n_2) = \nu_p(n_1) + \nu_p(n_2) \quad \text{and} \quad \nu_p(n_1 + n_2) \geq \min\{\nu_p(n_1), \nu_p(n_2)\}.$$

If $\nu_p(n_1) \neq \nu_p(n_2)$ then the inequality can be replaced by the equality.

Let

$$f(x)=\sum_{n=0}^{\infty}a_nx^n\in\mathbb{Z}[[x]]$$

and

$$g(x)=\sum_{n=0}^{\infty}b_nx^n\in\mathbb{Z}[[x]]$$

be a formal power series with integer coefficients and $M\in\mathbb{N}_{\geq 2}$ be given. We say that f, g are *congruent modulo* M if and only if for all n the coefficients of x^n in both series are congruent modulo M.

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In other words

$$f\equiv g \pmod{M} \Longleftrightarrow \forall n\in\mathbb{N}:\ a_n\equiv b_n \pmod{M}.$$

One can prove that for any given $f, F, g, G \in \mathbb{Z}[[x]]$ satisfying

$$f \equiv g \pmod{M} \quad \text{and} \quad F \equiv G \pmod{M}$$

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Moreover, if $f(0), g(0) \in \{-1, 1\}$ then the series $1/f, 1/g$ have integer coefficients and we also have

$$\frac{1}{f} \equiv \frac{1}{g} \pmod{M}.$$

In consequence, in this case we have

$$f^k \equiv g^k \pmod{M}$$

for any $k \in \mathbb{Z}$.

We formulate the following general

Question 1

Let $f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n\in\mathbb{Z}[[x]]$ *with* $\varepsilon_0\in\{-1,1\}$ *and take* $m\in\mathbb{N}_+$. *What can be said about the sequences* $(\nu_p(a_m(n)))_{n\in\mathbb{N}},(\nu_p(b_m(n)))_{n\in\mathbb{N}}$, *where*

$$f(x)^m=\left(\sum_{n=0}^{\infty}\varepsilon_n x^n\right)^m=\sum_{n=0}^{\infty}a_m(n)x^n,$$

$$\frac{1}{f(x)^m}=\left(\frac{1}{\sum_{n=0}^{\infty}\varepsilon_n x^n}\right)^m=\sum_{n=0}^{\infty}b_m(n)x^n,$$

i.e., $a_m(n)$ $(b_m(n))$ *is the n-th coefficient in the power series expansion of the series* $f^m(x)$ $(1/f(x)^m$ *respectively)?*

It is clear that in its full generality, the Question 1 is too difficult and we cannot expect that the sequences $(\nu_p(a_m(n)))_{n\in\mathbb{N}}$ $(\nu_p(b_m(n)))_{n\in\mathbb{N}}$ can be given in closed form or even that a reasonable description can be obtained. Indeed, in order to give an example let us consider the formal power series

$$f(x)=\prod_{n=1}^{\infty}(1-x^n)=1+\sum_{n=1}^{\infty}(-1)^n(x^{\frac{n(3n-1)}{2}}+x^{\frac{n(3n+1)}{2}}).$$

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In particular $a(n)\in\{-1,0,1\}$ and thus for any given $p\in\mathbb{P}$ we have $\nu_p(a(n))=0$ in case when n is of the form $n=\frac{m(3m\pm1)}{2}$ for some $m\in\mathbb{N}_+$, and $\nu_p(a(n))=\infty$ in the remaining cases.

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However, the characterization of the 2-adic behaviour of the sequence $(p(n))_{n\in\mathbb{N}}$ given by

$$\frac{1}{f(x)}=\prod_{n=1}^{\infty}\frac{1}{1-x^n}=1+\sum_{n=1}^{\infty}p(n)x^n$$

is unknown. Let us note that the number $p(n)$ counts the integer partitions of n, i.e., the number of non-negative integer solutions of the equation $\sum_{i=1}^{n}x_i=n$. In fact, even the proof that $\nu_2(p(n))>0$ infinitely often is quite difficult.

The Prouhet-Thue-Morse sequence and the binary partition function

Let $n \in \mathbb{N}$ and $n = \sum_{i=0}^{k} \varepsilon_i 2^i$ be the unique expansion of n in base 2 and define the sum of digits function

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$$s_2(n) = \sum_{i=0}^{k} \varepsilon_i.$$

Next, we define the Prouhet-Thue-Morse sequence $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$ (on the alphabet $\{-1, +1\}$) in the following way

$$t_n = (-1)^{s_2(n)},$$

i.e., $t_n = 1$ if the number of 1's in the binary expansion of n is even and $t_n = -1$ in the opposite case. We will call the sequence $\mathbf{t}$ as the PTM sequence in the sequel.

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From the relations

$$s_2(0) = 0, \quad s_2(2n) = s_2(n), \quad s_2(2n+1) = s_2(n) + 1$$

we deduce the recurrence relations for the PTM sequence: $t_0 = 1$ and

$$t_{2n} = t_n, \quad t_{2n+1} = -t_n.$$

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$$T(x)=\prod_{n=0}^{\infty}(1-x^{2^n}).$$

Let us also note that the (multiplicative) inverse of the series T, i.e.,

$$B(x)=\frac{1}{T(x)}=\prod_{n=0}^{\infty}\frac{1}{1-x^{2^n}}=\sum_{n=0}^{\infty}b_nx^n$$

is an interesting object.

Indeed, for $n \in \mathbb{N}$, the number b_n counts the number of binary partitions of n. The binary partition is the representation of the integer n in the form

$$n = \sum_{i=0}^{n} u_i 2^i,$$

where $u_i \in \mathbb{N}$ for $i = 0, \ldots, n$.

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The sequence $(b_n)_{n \in \mathbb{N}}$ was introduced by Euler. However, it seems that the first nontrivial result concerning its arithmetic properties was obtained by Churchhouse. He proved that $\nu_2(b_n) \in \{1, 2\}$ for $n \geq 2$.

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More precisely, $b_0 = 1, b_1 = 1$ and for $n \geq 2$ we have $\nu_2(b_n) = 2$ if and only if n or $n - 1$ can be written in the form $4^r(2u + 1)$ for some $r \in \mathbb{N}_+$ and $u \in \mathbb{N}$. In the remaining cases we have $\nu_2(b_n) = 1$.

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We can compactly write

$$\nu_2(b_n) = \begin{cases} \frac{1}{2}|t_n - 2t_{n-1} + t_{n-2}|, & \text{if } n \geq 2 \\ 0, & \text{if } n \in \{0, 1\}. \end{cases}$$

In other words we have simple characterization of the 2-adic valuation of the number b_n for all $n \in \mathbb{N}$.

Let $m \in \mathbb{N}_+$ and consider the series

$$B_m(x) := B(x)^m = \prod_{n=0}^{\infty} \frac{1}{(1-x^{2^n})^m} = \sum_{n=0}^{\infty} b_m(n)x^n.$$

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We have $b_1(n) = b_n$ for $n \in \mathbb{N}$ and

$$b_m(n) = \sum_{i_1+i_2+\ldots+i_m=n} \prod_{k=1}^{m} b(i_k),$$

i.e., $b_m(n)$ is Cauchy convolution of m-copies of the sequence $(b_n)_{n\in\mathbb{N}}$. For $m \in \mathbb{N}_+$ we denote the sequence $(b_m(n))_{n\in\mathbb{N}}$ by $\mathbf{b}_m$.

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From the above expression we easily deduce that the number $b_m(n)$ has a natural combinatorial interpretation. Indeed, $b_m(n)$ counts the number of representations of the integer n as the sum of powers of 2, where each summand can have one of m colors.

Now we can formulate the natural

Question 2

Let $m \in \mathbb{N}_+$ be given. What can be said about the sequence $(\nu_2(b_m(n)))_{n\in\mathbb{N}}$?

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Lemma 1

Let $m \in \mathbb{N}_+$ be fixed and write $m = 2^k(2u+1)$ with $k \in \mathbb{N}$. Then:

1. *We have $b_m(n) \equiv \binom{m}{n} + 2^{k+1}\binom{m-2}{n-2} \pmod{2^{k+2}}$ for m even;*
2. *We have $b_m(n) \equiv \binom{m}{n} \pmod{2}$ for m odd;*
3. *For infinitely many n we have $b_m(n) \not\equiv 0 \pmod{4}$ for m odd.*

Lemma 2

Let m be a positive integer ≥ 2*. Then*

$$\binom{2^m-1}{k} \equiv 1 \pmod{2}, \quad \textit{for} \quad k = 0, 1, \ldots, 2^m - 1,$$

and

$$\binom{2^m}{k} \equiv \begin{cases} 1 & \textit{for } k = 0, 2^m \\ 4 & \textit{for } k = 2^{m-2}, 3 \cdot 2^{m-2} \\ 6 & \textit{for } k = 2^{m-1} \\ 0 & \textit{ in the remaining cases} \end{cases} \pmod{8}, \quad \textit{for} \quad k = 0, 1, \ldots, 2^m.$$

We are ready to prove the following

Theorem 3

Let $k \in \mathbb{N}_+$ be given. Then $\nu_2(b_{2^k-1}(n)) = 0$ for $n \leq 2^{k-1}$ and

$$\nu_2(b_{2^k-1}(2^k n + i)) = \nu_2(b_1(2n))$$

for each $i \in \{0, \ldots, 2^k - 1\}$ and $n \in \mathbb{N}_+$.

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Proof: First of all, let us observe that the second part of Lemma 1 and the first part of Lemma 2 implies that $b_{2^k-1}(n)$ is odd for $n \leq 2^k - 1$ and thus $\nu_2(b_{2^k-1}(n)) = 0$ in this case.

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Let us observe that from the identity $B_{2^k-1}(x) = T(x)B_{2^k}(x)$ we get the identity

$$b_{2^k-1}(n) = \sum_{j=0}^{n} t_{n-j} b_{2^k}(j), \tag{1}$$

where t_n is the n-th term of the PTM sequence.

Now let us observe that from the first part of Lemma 1 and the second part of Lemma 2 we have

$$b_{2^k}(n) \equiv \binom{2^k}{n} \pmod{8}$$

for $n = 0, 1, \ldots, 2^k$ and $b_{2^k}(n) \equiv 0 \pmod{8}$ for $n > 2^k$, provided $k \geq 2$ or $n \neq 2$.

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Moreover,

$$b_2(2) \equiv \binom{2}{2} + 4\binom{0}{0} = 5 \pmod 8.$$

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Moreover,

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Summing up this discussion we have the following expression for $b_{2^k-1}(n) \pmod{8}$, where $k \geq 2$ and $n \geq 2^k$:

$$\begin{aligned} b_{2^k-1}(n) &= \sum_{j=0}^{n} t_{n-j} b_{2^k}(j) = \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) + \sum_{j=2^k+1}^{n} t_{n-j} b_{2^k}(j) \\ &\equiv \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) \equiv \sum_{j=0}^{2^k} t_{n-j} \binom{2^k}{j} \pmod{8} \\ &\equiv t_n + t_{n-2^k} + 4t_{n-2^{k-2}} + 4t_{n-3\cdot 2^{k-2}} + 6t_{n-2^{k-1}} \pmod{8}. \end{aligned}$$

However, it is clear that $t_{n-2^{k-2}} + t_{n-3\cdot 2^{k-2}} \equiv 0 \pmod{2}$ and thus we can simplify the above expression and get

$$b_{2^k-1}(n) \equiv t_n + t_{n-2^k} + 6t_{n-2^{k-1}} \pmod{8}$$

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for $n \geq 2^k$.

If $k = 1$ and $n \geq 2$ then, analogously, we get

$$b_1(n) \equiv \sum_{j=0}^{2^k} t_{n-j} b_{2^k}(j) \pmod{8} \equiv t_n + 5t_{n-2} + 2t_{n-1} \pmod{8}$$

and since $t_{n-1} \equiv t_{n-2} \pmod{2}$, we thus conclude that

$$b_1(n) \equiv t_n + t_{n-2^k} + 6t_{n-2^{k-1}} \pmod{8}.$$

Let us put

$$R_k(n) = t_n + t_{n-2^k} + 6t_{n-2^{k-1}}.$$

Using now the recurrence relations for t_n, i.e., $t_{2n} = t_n, t_{2n+1} = -t_n$, we easily deduce the identities

$$R_k(2n) = R_{k-1}(n), \quad R_k(2n+1) = -R_{k-1}(n)$$

for $k \geq 2$.

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for $k \geq 2$.

Using a simple induction argument, one can easily obtain the following identities:

$$|R_k(2^k m + j)| = |R_1(2m)| \tag{2}$$

for $k \geq 2, m \in \mathbb{N}$ and $j \in \{0, \ldots, 2^k - 1\}$.

From the above identity we easily deduce that $R_k(n) \not\equiv 0 \pmod 8$ for each $n \in \mathbb{N}$ and each $k \geq 1$. If $k = 1$ then $R_1(n) = t_n + 6t_{n-1} + t_{n-2}$ and $R_1(n) \equiv 0 \pmod 8$ if and only if $t_n = t_{n-1} = t_{n-2}$. However, a well known property of the Prouhet-Thue-Morse sequence is that there are no three consecutive terms which are equal.

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If $k \geq 2$ then our statement about $R_k(n)$ is clearly true for $n \leq 2^k$. If $n > 2^k$ then we can write $n = 2^k m + j$ for some $m \in \mathbb{N}$ and $j \in \{0, 1, \ldots, 2^k - 1\}$. Using the reduction (2) and the property obtained for $k = 1$, we get the result.

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Summing up our discussion, we have proved that $\nu_2(b_{2^k-1}(n)) \leq 2$ for each $n \in \mathbb{N}$, since $\nu_2(b_1(n)) \in \{0, 1, 2\}$. Moreover, as an immediate consequence of our reasoning we get the equality

$$\nu_2(b_{2^k-1}(2^k n + j)) = \nu_2(b_1(2n))$$

for $j \in \{0, ..., 2^k - 1\}$ and our theorem is proved.

Conjecture 1

Let $m \in \mathbb{N}_{\geq 2}$ *be given and suppose that* m *is not of the form* $2^k - 1$ *for* $k \in \mathbb{N}_+$. *Then the sequence* $(\nu_2(b_m(n)))_{n\in\mathbb{N}}$ *is unbounded.*

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Conjecture 2

Let m be a fixed positive integer. Then for each $n \in \mathbb{N}$ and $k \geq m+2$ the following congruence holds

$$b_{2^m}(2^{k+1}n) \equiv b_{2^m}(2^{k-1}n) \pmod{2^k}.$$

Conjecture 3

Let m be a fixed positive integer. Then for each $n \in \mathbb{N}$ and $k \geq m+2$ the following congruence holds

$$b_{2^m-1}(2^{k+1}n) \equiv b_{2^m-1}(2^{k-1}n) \pmod{2^{4\lfloor\frac{k+1}{2}\rfloor-2}}.$$

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In fact we expect the following

Conjecture 4

Let m be a fixed positive integer. Then for each $n \in \mathbb{N}$ and given $k \gg 1$ there is a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(k) = O(k)$ and the following congruence holds

$$b_m(2^{k+1}n) \equiv b_m(2^{k-1}n) \pmod{2^{f(k)}}.$$

Some general results

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of integers and write $f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n\in\mathbb{Z}[[x]]$. Moreover, for $m\in\mathbb{N}_+$ we define the sequence $\mathbf{b}_m=(b_m(n))_{n\in\mathbb{N}}$, where

$$\frac{1}{f(x)^m}=\sum_{n=0}^{\infty}b_m(n)x^n.$$

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$$\frac{1}{f(x)^m}=\sum_{n=0}^{\infty}b_m(n)x^n.$$

We have the following

Theorem 4

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of integers and suppose that $\varepsilon_n\equiv 1 \pmod{2}$ for each $n\in\mathbb{N}$. Then for any $m\in\mathbb{N}_+$ and $n\geq m$ we have the congruence

$$b_{m-1}(n)\equiv\sum_{i=0}^{m}\binom{m}{i}\varepsilon_{n-i} \pmod{2^{\nu_2(m)+1}}. \qquad (3)$$

Proof: Let $f(x)=\sum_{n=0}^{\infty}\varepsilon_n x^n\in\mathbb{Z}[[x]]$. From the assumption on sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we get that

$$f(x)\equiv\frac{1}{1+x}\pmod{2}.$$

In consequence, writing $m=2^{\nu_2(m)}k$ with k odd, and using the well known property saying that $U\equiv V\pmod{2^k}$ implies $U^2\equiv V^2\pmod{2^{k+1}}$, we get the congruence

$$\frac{1}{f(x)^m}\equiv(1+x)^m\pmod{2^{\nu_2(m)+1}}.$$

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$$\frac{1}{f(x)^m}\equiv(1+x)^m\pmod{2^{\nu_2(m)+1}}.$$

Thus, multiplying both sides of the above congruence by $f(x)$ we get

$$\frac{1}{f(x)^{m-1}}\equiv f(x)(1+x)^m\pmod{2^{\nu_2(m)+1}}.$$

From the power series expansion of $f(x)(1+x)^m$ by comparing coefficients on the both sides of the above congruence we get that

$$b_{m-1}(n)\equiv\sum_{i=0}^{\min\{m,n\}}\binom{m}{i}\varepsilon_{n-i}\pmod{2^{\nu_2(m)+1}},$$

i.e., for $n\geq m$ we get the congruence (3). Our theorem is proved.

From our result we can deduce the following

Corollary 5

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ *be a non-eventually constant sequence,* $\varepsilon_n \in \{-1,1\}$ *for each* $n\in\mathbb{N}$*, and suppose that for each* $N\in\mathbb{N}_+$ *there are infinitely many* $n\in\mathbb{N}$ *such that* $\varepsilon_n=\varepsilon_{n+1}=\ldots=\varepsilon_{n+N}$*. Then, for each even* $m\in\mathbb{N}_+$ *there are infinitely many* $n\in\mathbb{N}$ *such that*

$$\nu_2(b_{m-1}(n)) \geq \nu_2(m)+1 \quad \textit{and} \quad \nu_2(b_{m-1}(n+1))=1.$$

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Proof: From our assumption on the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we can find infinitely many $(m+1)$-tuples such that $\varepsilon_{n+1}=\varepsilon, \varepsilon_n=\ldots=\varepsilon_{n-m}=-\varepsilon$, where ε is a fixed element of $\{-1,1\}$. We apply (3) and get

$$b_{m-1}(n)\equiv\sum_{i=0}^{m}\binom{m}{i}\varepsilon_{n-i}\equiv-\sum_{i=0}^{m}\binom{m}{i}\varepsilon\equiv-\varepsilon 2^m\equiv 0 \pmod{2^{\nu_2(m)+1}},$$

$$b_{m-1}(n+1)\equiv\sum_{i=0}^{m}\binom{m}{i}\varepsilon_{n+1-i}\equiv 2\varepsilon-\sum_{i=0}^{m}\binom{m}{i}\varepsilon\equiv\varepsilon(2-2^m)\equiv 2\varepsilon \pmod{2^{\nu_2(m)+1}}.$$

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$$\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1 \quad \text{and} \quad \nu_2(b_{m-1}(n+1)) = 1.$$

Proof: From our assumption on the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we can find infinitely many $(m+1)$-tuples such that $\varepsilon_{n+1} = \varepsilon, \varepsilon_n = \ldots = \varepsilon_{n-m} = -\varepsilon$, where ε is a fixed element of $\{-1, 1\}$. We apply (3) and get

$$b_{m-1}(n) \equiv \sum_{i=0}^{m} \binom{m}{i} \varepsilon_{n-i} \equiv -\sum_{i=0}^{m} \binom{m}{i} \varepsilon \equiv -\varepsilon 2^m \equiv 0 \pmod{2^{\nu_2(m)+1}},$$

$$b_{m-1}(n+1) \equiv \sum_{i=0}^{m} \binom{m}{i} \varepsilon_{n+1-i} \equiv 2\varepsilon - \sum_{i=0}^{m} \binom{m}{i} \varepsilon \equiv \varepsilon(2 - 2^m) \equiv 2\varepsilon \pmod{2^{\nu_2(m)+1}}.$$

In consequence $\nu_2(b_{m-1}(n)) \geq \nu_2(m) + 1$ and $\nu_2(b_{m-1}(n+1)) = 1$ and our theorem is proved.

Example: Let $F:\ \mathbb{N}\rightarrow\mathbb{N}$ satisfy the condition $\limsup_{n\rightarrow+\infty}(F(n+1)-F(n))=+\infty$ and define the sequence

$$\varepsilon_n(F)=\begin{cases}1 & n=F(m) \text{ for some } m\in\mathbb{N}\\ -1 & \text{otherwise}\end{cases}.$$

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It is clear that the sequence $(\varepsilon_n(F))_{n\in\mathbb{N}}$ satisfies the conditions from Theorem 5 and thus for any even $m\in\mathbb{N}_+$ there are infinitely many $n\geq m$ such that $\nu_2(b_{m-1}(n))\geq\nu_2(m)+1$ and $\nu_2(b_{m-1}(n+1))=1$.

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A particular examples of F's satisfying required properties include:

- positive polynomials of degree ≥ 2;
- the functions which for given $n \in \mathbb{N}_+$ take as value the n-th prime number of the form $ak + b$, where $a \in \mathbb{N}_+, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$;
- and many others.

Lemma 6

Let $s \in \mathbb{N}_{\geq 3}$. *Then*

$$\binom{2^s}{i} \pmod{16} \equiv \begin{cases} 1 & \text{for } i = 0, 2^s \\ 6 & \text{for } i = 2^{s-1} \\ 8 & \text{for } i = (2j+1)2^{s-3}, j \in \{0,1,2,3\} \\ 12 & \text{for } i = 2^{s-2}, 3 \cdot 2^{s-2} \\ 0 & \text{in the remaining cases} \end{cases}.$$

Theorem 7

Let $s \in \mathbb{N}_+$ *and* $(\varepsilon_n)_{n\in\mathbb{N}}$ *be an integer sequence and suppose that* $\varepsilon_n \equiv 1 \pmod{2}$ *for* $n \in \mathbb{N}$*.*

(A) *For* $n \geq 2^s$ *we have*

$$b_{2^s-1}(n) \equiv \varepsilon_n + 2\varepsilon_{n-2^{s-1}} + \varepsilon_{n-2^s} \pmod{4}. \tag{4}$$

In particular, if $\varepsilon_n \in \{-1,1\}$ *for all* $n \in \mathbb{N}$ *then:*

$$\nu_2(b_{2^s-1}(n)) > 1 \iff \varepsilon_n = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s} \text{ or } \varepsilon_n = -\varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$$

$$\nu_2(b_{2^s-1}(n)) = 1 \iff \varepsilon_n = -\varepsilon_{n-2^s}.$$

(B) *For* $s \geq 2$ *and* $n \geq 2^s$ *we have*

$$b_{2^s-1}(n) \equiv \varepsilon_n + 6\varepsilon_{n-2^{s-1}} + \varepsilon_{n-2^s} \pmod{8}. \tag{5}$$

In particular, if $\varepsilon_n \in \{-1,1\}$ *for all* $n \in \mathbb{N}$*, then:*

$$\nu_2(b_{2^s-1}(n)) > 2 \iff \varepsilon_n = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$$

$$\nu_2(b_{2^s-1}(n)) = 2 \iff \varepsilon_n = -\varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$$

$$\nu_2(b_{2^s-1}(n)) = 1 \iff \varepsilon_n = -\varepsilon_{n-2^s}.$$

Theorem 7 (continuation)

(C) *For $s \geq 3$ and $n \geq 2^s$ we have*

$$b_{2^s-1}(n) \equiv \varepsilon_n + \varepsilon_{n-2^s} + 6\varepsilon_{n-2^{s-1}} + 12(\varepsilon_{n-2^{s-2}} + \varepsilon_{n-3\cdot 2^{s-2}}) \pmod{16} \quad (6)$$

In particular, if $\varepsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$, then:

$$\begin{aligned}
\nu_2(b_{2^s-1}(n)) > 3 \quad &\iff \quad \varepsilon_n = \varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \quad \text{or} \\
&\qquad\quad \varepsilon_n = -\varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = -\varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s}; \\
\nu_2(b_{2^s-1}(n)) = 3 \quad &\iff \quad \varepsilon_n = \varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = -\varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \quad \text{or} \\
&\qquad\quad \varepsilon_n = -\varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \\
&\iff \quad \varepsilon_n \equiv -\varepsilon_{n-2^s} + 2\varepsilon_{n-2^{s-1}} + 8 \pmod{16}
\end{aligned} \quad (7)$$

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In particular, if $\varepsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$, then:

$$\begin{array}{lcl} \nu_2(b_{2^s-1}(n)) > 3 & \iff & \varepsilon_n = \varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \quad \text{or} \\ & & \varepsilon_n = -\varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = -\varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s}; \\ \nu_2(b_{2^s-1}(n)) = 3 & \iff & \varepsilon_n = \varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = -\varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \quad \text{or} \\ & & \varepsilon_n = -\varepsilon_{n-2^{s-2}} = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-3\cdot 2^{s-2}} = \varepsilon_{n-2^s} \\ & \iff & \varepsilon_n \equiv -\varepsilon_{n-2^s} + 2\varepsilon_{n-2^{s-1}} + 8 \pmod{16} \end{array} \quad (7)$$

As a first application of Theorem 17 we get the following:

Corollary 8

Let $s \in \mathbb{N}_{\geq 2}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in \{-1, 1\}$ for all $n \in \mathbb{N}$. If there is no $n \in \mathbb{N}_{\geq 2^s}$ such that $\varepsilon_n = \varepsilon_{n-2^{s-1}} = \varepsilon_{n-2^s}$ then

$$\nu_2(b_{2^s-1}(n)) = \nu_2(\varepsilon_n + 6\varepsilon_{n-2^{s-1}} + \varepsilon_{n-2^s}).$$

In particular, for each $n \in \mathbb{N}_{\geq 2^s}$ we have $\nu_2(b_{2^s-1}(n)) \in \{1, 2\}$.

2-adic valuations for all powers

We consider now the power series

$$F_1(x) = \frac{1}{1-x}\prod_{n=0}^{\infty}\frac{1}{1-x^{2^n}} = \sum_{n=0}^{\infty} b_{2n}x^n,$$

where b_n is the binary partition function.

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where b_n is the binary partition function.

Let $m \in \mathbb{Z}$ and write

$$F_m(x) = F_1(x)^m = \frac{1}{(1-x)^m}\prod_{n=0}^{\infty}\frac{1}{(1-x^{2^n})^m} = \sum_{n=0}^{\infty} c_m(n)x^n.$$

If $m \in \mathbb{N}_+$, then the sequence $(c_m(n))_{n\in\mathbb{N}}$, has a natural combinatorial interpretation. More precisely, the number $c_m(n)$ counts the number of binary representations of n such that the part equal to 1 can take one among $2m$ colors and other parts can have m colors. Motivated by the mentioned result concerning the 2-adic valuation of the number $b_m(n)$, it is natural to ask about the behaviour of the sequence $(\nu_2(c_m(n))_{n\in\mathbb{N}}, m \in \mathbb{Z}$.

Let us observe the identity $F_1(x) = \frac{1}{1-x}B(x)$. Thus, the functional relation $(1-x)B(x) = B(x^2)$ implies the functional relation $(1-x)F_1(x) = (1+x)F_1(x^2)$ for the series F_1. In consequence, for $m \in \mathbb{Z}$ we have the relation

$$F_m(x) = \left(\frac{1+x}{1-x}\right)^m F_m(x^2),$$

which will be useful later.

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In the sequel we will need the following functional property: for $m_1, m_2 \in \mathbb{Z}$ we have

$$F_{m_1}(x)F_{m_2}(x) = F_{m_1+m_2}(x).$$

We start our investigations with the simple lemma which is a consequence of the result of Churchhouse and the product form of the series $F_{-1}(x)$.

Lemma 9

For $n \in \mathbb{N}_+$, we have the following equalities:

$$\nu_2(c_1(n)) = \frac{1}{2}|t_n + 3t_{n-1}|,$$

$$\nu_2(c_{-1}(n)) = \begin{cases} 1, & \text{if } t_n \neq t_{n-1} \\ +\infty, & \text{if } t_n = t_{n-1} \end{cases}.$$

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Proof: The first equality is an immediate consequence of the equalities $c_1(n) = b(2n), \nu_2(b(n)) = \frac{1}{2}|t_n - 2t_{n-1} + t_{n-2}|$ and the recurrence relations satisfied by the PTM sequence $(t_n)_{n\in\mathbb{N}}$, i.e., $t_{2n} = t_n, t_{2n+1} = -t_n$. The second equality comes from the expansion

$$F_{-1}(x) = (1-x)\prod_{n=0}^{\infty}(1-x^{2^n}) = (1-x)\sum_{n=0}^{\infty} t_n x^n = 1 + \sum_{n=1}^{\infty}(t_n - t_{n-1})x^n.$$

In order to compute the 2-adic valuations of the sequence $(c_{\pm 2}(n))_{n\in\mathbb{N}}$ we need the following simple

Lemma 10

The sequence $(c_{\pm 2}(n))_{n\in\mathbb{N}}$ *satisfy the following recurrence relations:* $c_{\pm 2}(0)=1, c_{\pm 2}(1)=\pm 4$ *and for* $n\geq 1$ *we have*

$$c_{\pm 2}(2n)=\pm\, 2c_{\pm 2}(2n-1)-c_{\pm 2}(2n-2)+c_{\pm 2}(n)+c_{\pm 2}(n-1),$$

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$$c_{\pm 2}(2n+1)=\pm\, 2c_{\pm 2}(2n)-c_{\pm 2}(2n-1)\pm 2c_{\pm 2}(n).$$

Proof: The recurrence relations for the sequence $(c_{\pm 2}(n))_{n\in\mathbb{N}}$ are immediate consequence of the functional equation $F_{\pm 2}(x)=\left(\frac{1+x}{1-x}\right)^{\pm 2}F_{\pm 2}(x^2)$, which can be rewritten in an equivalent form $(1-x)^{\pm 2}F_{\pm 2}(x)=(1+x)^{\pm 2}F_{\pm 2}(x^2)$. Comparing now the coefficients on both sides of this relation we get the result.

As a consequence of the recurrence relations for $(c_{\pm 2}(n))_{n\in\mathbb{N}}$ we get

Corollary 11

For $n \in \mathbb{N}_+$ *we have* $c_{\pm 2}(n) \equiv 4 \pmod 8$. *In consequence, for* $n \in \mathbb{N}_+$ *we have* $\nu_2(c_{\pm 2}(n)) = 2$.

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Proof: The proof relies on a simple induction. Indeed, we have $c_{\pm 2}(1) = \pm 4$, $c_{-2}(2) = 4$, $c_2(2) = 12$ and thus our statement folds for $n = 1, 2$. Assuming it holds for all integers $\leq n$ and applying the recurrence relations given in Lemma 10 we get the result.
The second part is an immediate consequence of the obtained congruence.

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Theorem 12

Let $m \in \mathbb{Z} \setminus \{0, -1\}$ and consider the sequence $\mathbf{c}_m = (c_m(n))_{n\in\mathbb{N}}$. Then $c_m(0) = 1$ and for $n \in \mathbb{N}_+$ we have

$$\nu_2(c_m(n)) = \begin{cases} \nu_2(m) + 1, & \text{if } m \equiv 0 \pmod 2 \\ 1, & \text{if } m \equiv 1 \pmod 2 \text{ and } t_n \neq t_{n-1} \\ \nu_2(m+1) + 1, & \text{if } m \equiv 1 \pmod 2 \text{ and } t_n = t_{n-1} \end{cases} . \tag{8}$$

Proof: First of all, let us note that our theorem is true for $m = 1, \pm 2$. This is a consequence of Lemma 9 and Corollary 11. Let $m \in \mathbb{Z}$ and $|m| > 2$. Because $c_m(0) = 1, c_m(1) = 2m$ our statement is clearly true for $n = 0, 1$. We can assume that $n \geq 2$.

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We start with the case $m = -3$. From the functional relation $F_{-3}(x) = F_{-2}(x)F_{-1}(x)$ we immediately get the identity

$$c_{-3}(n) = \sum_{n=0}^{n} c_{-1}(i)c_{-2}(n-i) = c_{-2}(n)+t_n-t_{n-1}+\sum_{i=1}^{n-1}(t_i-t_{i-1})c_{-2}(n-i).$$

Let us observe that for $i \in \{1, \ldots, n-1\}$, from Lemma 9 and Corollary 11, we obtain the inequality

$$\nu_2((t_i - t_{i-1})c_{-2}(n-i)) \geq 3.$$

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In consequence, from Lemma 10, we get

$$c_{-3}(n) \equiv c_{-2}(n) + t_n - t_{n-1} \equiv 4 + t_n - t_{n-1} \pmod{8}.$$

It is clear that $4 + t_n - t_{n-1} \not\equiv 0 \pmod{8}$. Thus, we get the equality $\nu_2(c_{-3}(n)) = \nu_2(4 + t_n - t_{n-1})$ and the result follows for $m = -3$.

We are ready to prove the general result. We proceed by double induction on m (which depends on the remainder of $m \pmod 4$) and $n \in \mathbb{N}_+$. As we already proved, our theorem is true for $m = \pm 1, \pm 2$ and $m = -3$. Let us assume that it is true for each m satisfying $|m| < M$ and each term $c_m(j)$ with $j < n$. Let $|m| \geq M$ and write $m = 4k + r$ with $|k| < M/4$ for some $r \in \{-3, -2, 0, 1, 2, 3\}$ (depending on the sign of m).

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If $m = 4k$, then from the identity $F_{4k}(x) = F_{2k}(x)^2$ we get the expression

$$c_{4k}(n) = 2c_{2k}(n) + \sum_{i=1}^{n-1} c_{2k}(i)c_{2k}(n-i).$$

From the induction hypothesis we have
$\nu_2(c_{2k}(i)c_{2k}(n-i)) = 2(\nu_2(2k)+1) > \nu_2(2c_{2k}(n)) = \nu_2(2k) + 2$. In consequence
$\nu_2(c_m(n)) = \nu_2(c_{4k}(n)) = \nu_2(2c_{2k}(n)) = \nu_2(2k) + 2 = \nu_2(4k) + 1$. The obtained equality finishes the proof in the case $m \equiv 0 \pmod 4$.

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Similarly, if $m = 4k+2$ is positive, we use the identity $F_{4k+2}(x) = F_{4k}(x)F_2(x)$, and get

$$c_{4k+2}(n) = c_2(n) + c_{4k}(n) + \sum_{i=1}^{n-1} c_{4k}(i)c_2(n-i).$$

From the equalities $\nu_2(c_2(n)) = \nu_2(2) + 1$ and $\nu_2(c_{4k}(n)) = \nu_2(4k) + 1, n \in \mathbb{N}_+$, we get $\nu_2(c_{4k}(i)c_2(n-i)) = \nu_2(k) + 5$ for each $i \in \{1, \ldots, n-1\}$. Thus $\nu_2(c_2(n) + c_{4k}(n)) = \nu_2(c_2(n)) = 2 = \nu_2(4k+2) + 1.$

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If $m = 4k + 2$ is negative, we use the identity $F_{4k+2}(x) = F_{4(k+1)}(x)F_{-2}(x)$ and proceed in exactly the same way.

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If $m = 4k + 2$ is negative, we use the identity $F_{4k+2}(x) = F_{4(k+1)}(x)F_{-2}(x)$ and proceed in exactly the same way.

If $m = 4k + 1 > 0$, then we use the identity $F_{4k+1}(x) = F_{4k}(x)F_1(x)$ and get

$$c_{4k+1}(n) = c_{4k}(n) + c_1(n) + \sum_{i=1}^{n-1} c_{4k}(i)c_1(n-i).$$

From induction hypothesis we have $\nu_2(c_{4k}(i)c_1(n-i)) \geq \nu_2(4k) + 2 \geq 4$. Moreover, for $n \in \mathbb{N}_+$ we have $\nu_2(c_1(n)) \in \{1, 2\}$. Thus

$$\nu_2(c_{4k}(n) + c_1(n)) = \nu_2(c_1(n)) = \begin{cases} 1, & \text{if } t_n \neq t_{n-1} \\ 2, & \text{if } t_n = t_{n-1} \end{cases}.$$

as we claimed.

If $m = 4k + 1 < 0$, we write $m = 4(k + 1) - 3$ and use the identity $F_{4k+1}(x) = F_{4(k+1)}(x)F_{-3}(x)$. Next, using the obtained expression for $\nu_2(c_{-3}(n))$ and $\nu_2(c_{4(k+1)}(n))$ and the same reasoning as in the positive case we get the result.

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It is clear that $\nu_2(c_{4k}(i)c_{-1}(n-i)) > \nu_2(c_{4k}(n) + c_{-1}(n))$ for each $n \in \mathbb{N}_+$ and $i \in \{1, \ldots, n-1\}$. In consequence, by induction hypothesis

$$\begin{aligned}\nu_2(c_{4k+3}(n)) &= \nu_2(c_{4(k+1)}(n) + c_{-1}(n)) \\ &= \begin{cases} 1, & \text{if } t_n \neq t_{n-1} \\ \nu_2(c_{4(k+1)}(n)), & \text{if } t_n = t_{n-1} \end{cases} \\ &= \begin{cases} 1, & \text{if } t_n \neq t_{n-1} \\ \nu_2(4k+3+1)+1, & \text{if } t_n = t_{n-1} \end{cases}.\end{aligned}$$

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If $m = 4k + 3 < 0$, then we write $4k + 3 = 4(k + 1) - 1$ and employ the identity $F_{4k+3}(x) = F_{4(k+1)}(x)F_{-1}(x)$.

Let $n \in \mathbb{N}_+$ and write

$$n = \sum_{i=0}^{k} \varepsilon_i 2^i,$$

where $\varepsilon_i \in \{0, 1\}$ and $k \leq \log_2 n$. The above representation is just the (unique) binary expansion of n in base 2. Let us observe that the equality $\nu_2(n) = u$ implies $\varepsilon_0 = \ldots = \varepsilon_{u-1} = 0$ and $\varepsilon_u = 1$ in the above representation. Thus, if $m \in \mathbb{Z} \setminus \{-1\}$ is fixed, our result concerning the exact value of $\nu_2(c_m(n))$ given by Theorem 16 implies that the number of trailing zeros in the binary expansion of $c_m(n), n \in \mathbb{N}_+$, is bounded.

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This observation suggests the question whether the index of the next non-zero digit in the binary expansion in $c_m(n)$ is in bounded distance from the first one. We state this in equivalent form as the following

Question 3

Does there exists $m \in \mathbb{Z} \setminus \{-1\}$ such that the sequence

$$\left(\nu_2\left(\frac{c_m(n)}{2^{\nu_2(c_m(n))}} - 1\right)\right)_{n \in \mathbb{N}}$$

has finite set of values?

Let us write $d_m(n) = \nu_2\left(\frac{c_m(n)}{2^{\nu_2(c_m(n))}} - 1\right)$. We performed numerical computations for $m \in \mathbb{Z}$ satisfying $|m| < 100$ and $n \leq 10^5$. In this range there are many values of m such that the cardinality of the set of values of the sequence $(d_m(n))_{n\in\mathbb{N}}$ is ≤ 4. We define:

$$M_m(x) := \max\{d_m(n) : \; n \leq x\}, \quad L_m(x) := |\{d_m(n) : \; n \leq x\}|.$$

m	$M_m(10^5)$	$L_m(10^5)$		m	$M_m(10^5)$	$L_m(10^5)$
-97	5	3		3	2	2
-93	2	2		15	4	3
-89	3	3		23	3	3
-81	4	3		27	2	2
-69	2	2		35	2	2
-65	6	4		39	3	3
-61	2	2		47	4	3
-49	4	3		59	2	2
-41	3	3		63	6	4
-37	2	2		67	2	2
-29	2	2		79	4	3
-25	3	3		87	3	3
-17	4	3		91	2	2
-5	2	2		95	5	3
				99	2	2

Our numerical computations strongly suggest that there should be infinitely many $m \in \mathbb{Z}$ such that the sequence $(d_m(n))_{n\in\mathbb{N}}$ is bounded. We even dare to formulate the following

Conjecture 5

Let $k \in \mathbb{N}_+$ and $m = 2^{2^k} - 1$. Then the sequence $(d_m(n))_{n\in\mathbb{N}}$ is bounded.

In fact, we expect that for $n \in \mathbb{N}$ the inequality $d_{2^{2^k}-1}(n) \leq 2k$ is true.

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It is well known that if $k \in \mathbb{N}_+$ and $t \equiv 1 \pmod{2}$, then

$$c_1(2^{2^{k+1}}t) - c_1(2^{2^{k-1}}t) \equiv 0 \pmod{2^{3k+2}},$$
$$c_1(2^{2^k}t) - c_1(2^{2^{k-2}}t) \equiv 0 \pmod{2^{3k}}$$

(remember $c_1(n) = b(2n)$, where $b(n)$ counts the binary partitions of n). The above congruences were conjectured by Churchhouse and independently proved by Rödseth and Gupta. Moreover, there is no higher power of 2 which divides $c_1(4n) - c_1(n)$.

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This result motivates the question concerning the divisibility of the number $c_m(2^{k+2}n) - c_m(2^k n)$ by powers of 2. We performed some numerical computations in case of $m \in \{2, 3, \ldots, 10\}$ and $n \leq 10^5$ and believe that the following is true.

Conjecture 6

For $k \in \mathbb{N}_+$ and each $n \in \mathbb{N}_+$, we have:

$$\nu_2(c_{2k}(4n) - c_{2k}(n)) = \nu_2(n) + 2\nu_2(k) + 3.$$

Moreover, for $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$ the following inequalities holds

$$\nu_2(c_{4k+1}(4n) - c_{4k+1}(n)) \geq \nu_2(n) + 3,$$
$$\nu_2(c_{4k+3}(4n) - c_{4k+3}(n)) \geq \nu_2(n) + 6.$$

In each case the equality holds for infinitely many $n \in \mathbb{N}$.

Some results for p-ary colored partitions

Some results for p-ary colored partitions

For $k \in \mathbb{N}_+$ we define the sequence $(A_{m,k}(n))_{n\in\mathbb{N}}$, where

$$F_m(x)^k = \prod_{n=0}^{\infty} \frac{1}{(1-x^{m^n})^k} = \sum_{n=0}^{\infty} A_{m,k}(n)x^n.$$

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The sequence $(A_{m,k}(n))_{n\in\mathbb{N}}$, as the sequences considered earlier, can be interpreted in a natural combinatorial way. More precisely, the number $A_{m,k}(n)$ counts the number of representations of n as sums of powers of m, where each summand has one among k colors.

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A question arises: is it possible to find a simple expression for an exponent k, such that the sequence $(\nu_p(A_{p,k}(n)))_{n\in\mathbb{N}}$ is bounded or even can be described in simple terms? Here p is a fixed prime number.

For a given p (non-necessarily a prime), an integer n and $i \in \{0, \ldots, p-1\}$ we define

$$N_p(i,n) = |\{j : \ n = \sum_{j=0}^{k} \varepsilon_j p^j, \varepsilon_j \in \{0, \ldots, p-1\} \text{ and } \varepsilon_j = i\}|.$$

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The above number counts the number of the digits equal to i in the base p representation of the integer n. From the definition, we immediately deduce the following equalities:

$$N_p(i,0) = 0, \quad N_p(i, pn+j) = \begin{cases} N_p(i,n), & \text{if } j \neq i \\ N_p(i,n)+1, & \text{if } j = i \end{cases}. \tag{9}$$

We have the following result

Lemma 13

Let $r \in \{1, \dots, p-1\}$. *We have*

$$F_p(x)^{-r} = \prod_{n=0}^{\infty}(1 - x^{p^n})^r = \sum_{n=0}^{\infty} D_{p,r}(n)x^n,$$

where

$$D_{p,r}(n) = \prod_{i=0}^{p-1}(-1)^{iN_p(i,n)}\binom{r}{i}^{N_p(i,n)}, \qquad (10)$$

with the convention that $\binom{a}{b} = 0$ *for* $b > a$ *and* $0^0 = 1$. *Moreover, for* $j \in \{0, \dots, p-1\}$ *and* $n \in \mathbb{N}_+$ *we have*

$$D_{p,r}(pn + j) = (-1)^j \binom{r}{j} D_{p,r}(n).$$

Our next result is the following

Lemma 14

Let $k \in \mathbb{N}_+$ *and suppose that* $p-1|k$. *Then*

$$F_p(x)^k \equiv (1-x)^{\frac{k}{p-1}} \pmod{p^{\nu_p(k)+1}}.$$

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We are ready to present the crucial lemma which is the main tool in our study of the p-adic valuation of the number $A_{p,(p-1)(up^s-1)}(n)$ in the sequel. More precisely, the lemma contains information about behaviour of the p-adic valuation of the expression

$$\sum_{i=0}^{u}(-1)^i\binom{u}{i}D_p(n-i),$$

where

$$D_p(n) := D_{p,p-1}(n).$$

In particular $D_p(n) \neq 0$ for all $n \in \mathbb{N}$.

Lemma 15

Let $p \geq 3$ be prime and $u \in \{1, \dots, p-1\}$. Let $n \geq p$ be of the form $n = n'' p^{s+1} + kp^s + j$ for some $n'' \in \mathbb{N}, k \in \{1, \dots, p-1\}, s \in \mathbb{N}_+$ and $j \in \{0, \dots, p-1\}$. Then the following equality holds:

$$\nu_p\left(\sum_{i=0}^{u}(-1)^i\binom{u}{i}D_p(n-i)\right) = \nu_p\left((p-k)\binom{p+u-1}{j} + k\binom{p+u-1}{p+j}\right).$$

In particular:

(a) *If $u = 1$, then*

$$\nu_p\left(\sum_{i=0}^{u}(-1)^i\binom{u}{i}D_p(n-i)\right) = \nu_p(D_p(n) - D_p(n-1)) = 1,$$

for any $n \in \mathbb{N}_+$.

(b) *If $j \geq u$, then we have the equality*

$$\nu_p\left(\sum_{i=0}^{u}(-1)^i\binom{u}{i}D_p(n-i)\right) = 1.$$

(c) *If $u \geq 2$, then there exist $j, k \in \{0, \dots, p-1\}, k \neq 0$, such that we have*

$$\nu_p\left(\sum_{i=0}^{u}(-1)^i\binom{u}{i}D_p(n-i)\right) \geq 2.$$

Theorem 16

Let $p \in \mathbb{P}_{\geq 3}$, $u \in \{1, \dots, p-1\}$ and $s \in \mathbb{N}_+$.

(a) *If $n > up^s$, then*
$$\nu_p(A_{p,(p-1)(up^s-1)}(n)) \geq 1.$$

(b) *If $n > p^s$, then*
$$\nu_p(A_{p,(p-1)(p^s-1)}(n)) = 1.$$

(c) *If $u \geq 2$, then*
$$\nu_p(A_{p,(p-1)(up^s-1)}(n)) = 1$$
for infinitely many n.

(d) *If $u \geq 2$, then*
$$\nu_p(A_{p,(p-1)(up^s-1)}(n)) \geq 2$$
for infinitely many n.

(e) *If $s \geq 2$ and $n \geq p^{s+1}$ with the unique base p-representation $n = \sum_{i=0}^{v} \varepsilon_i p^i$ and*
$$\nu_p(A_{p,(p-1)(up^s-1)}(n)) \in \{1, 2\},$$
then the value of $\nu_p(A_{p,(p-1)(up^s-1)}(n))$ depends only on the coefficient ε_s and the first non-zero coefficient ε_t with $t > s$.

(f) *If $s \geq 2$ and*
$$\nu_p(A_{p,(p-1)(up^s-1)}(n)) \leq s$$
for $n > up^s$, then also
$$\nu_p(A_{p,(p-1)(up^s-1)}(pn)) = \nu_p(A_{p,(p-1)(up^s-1)}(pn+i)) \text{ for } i = 1, 2, \dots, p-1.$$

In the opposite direction we have the following

Theorem 17

Let $k \in \mathbb{N}_+, p \in \mathbb{P}_{\geq 3}$ *and suppose that* $p^2(p-1)|k$ *and* $r \in \{1, \ldots, p-2\}$. *Then, there are infinitely many* $n \in \mathbb{N}_+$ *such that*

$$\nu_p(A_{p,k-r}(n)) \geq \nu_p(k).$$

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Our computational experiments suggests the following

Conjecture 7

Let $p \in \mathbb{P}_{\geq 3}$, $u \in \{2, \ldots, p-1\}$ and $s \in \mathbb{N}_+$. Then, for $n \geq up^s$ we have

$$\nu_p(A_{p,(p-1)(up^s-1)}(n)) \in \{1, 2\}.$$

Moreover, for each $n \in \mathbb{N}_+$ we have the equalities

$$\nu_p(A_{p,(p-1)(up^s-1)}(pn)) = \nu_p(A_{p,(p-1)(up^s-1)}(pn+i)),\ i = 1, \ldots, p-1.$$

Let $k \in \mathbb{N}_{\geq 2}$ be given. We say that the sequence $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ is k-automatic if and only if the following set

$$K_k(\varepsilon) = \{(\varepsilon_{k^i n+j})_{n\in\mathbb{N}} :\ i \in \mathbb{N} \text{ and } 0 \leq j < k^i\},$$

called the k-kernel of ε, is finite.

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In the case of $p = 2$ we know that the sequence $(\nu_2(A_{2,2^s-1}(n)))_{n\in\mathbb{N}}$ is 2-automatic (and it is not eventually periodic). In Theorem 16 we proved that the sequence $(\nu_p(A_{p,k}(n)))_{n\in\mathbb{N}}$ for $k = (p-1)(p^s-1)$ with $p \geq 3$, is eventually constant and hence k-automatic for any k.

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We calculated the first 10^5 elements of the sequence $(\nu_p(A_{p,(p-1)(up^s-1)}(n)))_{n\in\mathbb{N}}$ for any $p \in \{3,5,7\}, s \in \{1,2\}$ and $u \in \{1,\ldots,p-1\}$ and were not able to spot any general relations. Our numerical observations lead us to the following

Question 4

For which $p \in \mathbb{P}_{\geq 5}, s \in \mathbb{N}$ and $u \in \{2,\ldots,p-1\}$, the sequence $(\nu_p(A_{p,(p-1)(up^s-1)}(n)))_{n\in\mathbb{N}}$ is k-automatic for some $k \in \mathbb{N}_+$?

Finally, we formulate the following

Conjecture 8

Let $k \in \mathbb{N}_+, p \in \mathbb{P}$ *and suppose that* k *is not of the form* $(p-1)(up^s-1)$ *for* $s \in \mathbb{N}$ *and* $u \in \{1, \ldots, p-1\}$*. Then, the sequence* $(\nu_p(A_{p,k}(n)))_{n\in\mathbb{N}}$ *is unbounded.*

Thank you for your attention;-)