Matrix Representation for Multiplicative Nested Sums

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Acknowledgment

- Diane Shi

Objects

$$\mathcal{S}(f_1,\ldots,f_k;N,m) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1(n_1)\cdots f_k(n_k)$$

and

$$\mathcal{A}(f_1,\ldots,f_k;N,m) := \sum_{N>n_1>\cdots>n_k\geq m} f_1(n_1)\cdots f_k(n_k)$$

Example.

$$\left.\begin{array}{c} k=1 \\ m=1 \\ N=\infty \\ f_1(n)=\frac{1}{n^s} \end{array}\right\} \Rightarrow \mathcal{S}(f_1;\infty,1) = \mathcal{A}(f_1;\infty,1) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Objects

$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right)$$

and

$$\mathcal{A}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N>n_1>\cdots>n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right)$$

Example.

$$\left.\begin{array}{c} k=1 \\ m=1 \\ N=\infty \\ f_1(n)=\frac{1}{n^s} \end{array}\right\} \Rightarrow \mathcal{S}\left(f_1;\infty,1\right)=\mathcal{A}\left(f_1;\infty,1\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\zeta\left(s\right).$$

Objects

$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right)$$

and

$$\mathcal{A}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N> n_1>\cdots> n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right)$$

Let $m=1$, $N=\infty$, and $f_l(x)=1/x^{s_l}$ for $l=1,\ldots,k$. Then,

$$\mathcal{S}\left(f_1,\ldots,f_k;\infty,1\right) := \sum_{n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} = \zeta^*\left(s_1,\ldots,s_k\right),$$

and

$$\mathcal{A}\left(f_1,\ldots,f_k;\infty,1\right) := \sum_{n_1>\cdots> n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} = \zeta\left(s_1,\ldots,s_k\right).$$

An Important Example

Let $m = 1$ and

$$f_1(x) = \cdots = f_k(x) = \frac{1}{x}.$$

$$\mathcal{S}\left(\underbrace{\frac{1}{x}, \ldots \frac{1}{x}}_{k}; N, 1\right) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k}.$$

Theorem. [K. Dilcher]

$$\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = \sum_{\ell=1}^{N} (-1)^{\ell-1} \binom{N}{\ell} \frac{1}{\ell^k}.$$

An Important Example

Theorem. [K. Dilcher]

$$\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1\cdots n_k} = \sum_{\ell=1}^{N} (-1)^{\ell-1}\binom{N}{\ell}\frac{1}{\ell^k}.$$

Example. $k=2$, $N=3$:

$$\begin{aligned}\sum_{3\geq n_1\geq n_2\geq 1} &= \frac{1}{1\cdot 1}+\frac{1}{2\cdot 1}+\frac{1}{3\cdot 1}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2}+\frac{1}{3\cdot 3}\\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{9}\\ &= \frac{85}{36}\end{aligned}$$

$$\sum_{3\geq n_1\geq n_2\geq 1} = \sum_{\ell=1}^{3}(-1)^{\ell-1}\binom{3}{\ell}\frac{1}{\ell^2} = \frac{85}{36}$$

Outline

Random Walk and Matrix Representation

Diagonalization

Limits

Matrix

$$\mathbf{S}_N := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

Matrix

$$\mathbf{S}_N := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \Rightarrow \mathbf{S}_3 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\mathbf{S}_3^{2+1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{8} & \frac{1}{8} & 0 \\ \frac{85}{108} & \frac{19}{108} & \frac{1}{27} \end{pmatrix}$$

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$$\mathbf{S}_3^{2+1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{8} & \frac{1}{8} & 0 \\ \frac{85}{108} & \frac{19}{108} & \frac{1}{27} \end{pmatrix} \Rightarrow \frac{85}{36} = 3 \cdot \frac{85}{108} = 3 \cdot \left(\mathbf{S}_3^{2+1}\right)_{3,1}.$$

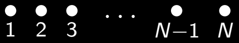
Theorem. [L. Jiu and D. Shi]

$$\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = N \cdot \left(\mathbf{S}_N^{k+1}\right)_{N,1}.$$

$$\sum_{2 \geq n_1 \geq n_2 \geq 1} \frac{1}{n_1 n_2} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{8}.$$

Random Walk

Label N sites as follows:



We start a random walk at site "N", with the rules:

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

$$\mathbb{P}(i \to j) = \text{the probability from site "}i\text{" to site "}j\text{".}$$

For example, suppose we are at site "6":

Then, the next step only allows to walk to sites $\{1, 2, 3, 4, 5, 6\}$,

$$\mathbb{P}(6 \to 6) = \mathbb{P}(6 \to 5) = \mathbb{P}(6 \to 4) = \mathbb{P}(6 \to 3) = \mathbb{P}(6 \to 2) = \mathbb{P}(6 \to 1) = \frac{1}{6}.$$

Random Walk (Continued)

Therefore, a typical walk is as follows:

STEP 1: $N \to n_1 \, (\leq N)$, with $\mathbb{P}\,(N \to n_1) = \frac{1}{N}$;

STEP 2: $n_1 \to n_2 \, (\leq n_1)$, with $\mathbb{P}\,(n_1 \to n_2) = \frac{1}{n_1}$;

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

STEP $k+1$: $n_k \to n_{k+1} \, (\leq n_k)$, with $\mathbb{P}\,(n_k \to n_{k+1}) = \frac{1}{n_k}$.

Focus on $\mathbb{P}\,(n_{k+1} = 1)$:

$$\mathbb{P}\,(n_{k+1} = 1) = \sum_{\text{All possible paths}} \frac{1}{N}\frac{1}{n_1}\cdots\frac{1}{n_k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k}.$$

Random Walk (Continued)

On the other hand, the transition matrix of sites $\{1, \ldots, N\}$ is exactly given by $\mathbf{S}_N$, i.e,

$$\mathbf{S}_N = (\alpha_{i,j}) \text{ with } \alpha_{i,j} = \mathbb{P}(i \to j) = \begin{cases} 1/i, & \text{if } j \le i; \\ 0, & \text{if } j > i. \end{cases}$$

Therefore, after $k+1$ steps, entries of $\mathbf{S}_N^{k+1}$ give the transition probabilities among sites. In particular,

$$\left(\mathbf{S}_N^{k+1}\right)_{N,1} = \mathbb{P}(n_{k+1} = 1) = \frac{1}{N} \sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1 \cdots n_k}.$$

Remark.

$$\frac{1}{N} \sum_{N \ge n_1 \ge \cdots \ge n_k \ge m} \frac{1}{n_1 \cdots n_k} = \mathbb{P}(n_{k+1} = m) = \left(\mathbf{S}_N^{k+1}\right)_{N,m}$$

Random Walk (Continued)

$$\mathbf{S}_N := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \Rightarrow \mathbf{S}_3 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\mathbf{S}_3^{2+1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{8} & \frac{1}{8} & 0 \\ \frac{85}{108} & \frac{19}{108} & \frac{1}{27} \end{pmatrix}$$

$$\sum_{3 \geq n_1 \geq n_2 \geq 2} = \frac{1}{4} + \frac{1}{6} + \frac{1}{9} = \frac{19}{36} = 3 \cdot \frac{19}{108}.$$

Main Results

Theorem. [L. Jiu and D. Shi] Define, for $l = 1, \ldots, k$,

$$\mathbf{P}_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

$$\mathbf{S}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix},$$

and

$$\mathbf{A}_{N|f_l} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f_l(1) & 0 & 0 & \cdots & 0 & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_l(N-1) & f_l(N-1) & f_l(N-1) & \cdots & f_l(N-1) & 0 \end{pmatrix}.$$

Main Results (Continued)

Then, it holds that

$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right)=\left(\mathbf{P}_N\cdot\prod_{l=1}^{k}\mathbf{S}_{N|f_l}\right)_{N,m},$$

and

$$\mathcal{A}\left(f_1,\ldots,f_k;N,m\right)=\left(\mathbf{P}_N.\prod_{l=1}^{k}\mathbf{A}_{N|f_l}\right)_{N,m}.$$

Proposition. [L. Jiu and D. Shi]

$$\mathbf{A}_{(N-1)|f}=\left(\delta_{i-1,j}\right)_{N\times N}\mathbf{S}_{N|f}.$$

If $f\left(1\right),\ldots,f\left(N\right)$ are distinct,

$$\mathbf{S}_{N|f}=\mathbf{D}_{N|f}\mathrm{diag}\left(f\left(1\right),\ldots,f\left(N\right)\right)\left(\mathbf{D}_{N|f}\right)^{-1},$$

where

$$\mathbf{D}_{N|f}=\left(a_{i,j}\right)_{N\times N}\quad\text{and}\quad\left(\mathbf{D}_{N|f}\right)^{-1}=\left(b_{i,j}\right)_{N\times N}$$

Main Results (Continued)

$$a_{i,j} := \frac{f(i)}{f(N)} \prod_{k=i+1}^{N} \left(1 - \frac{f(k)}{f(j)}\right),$$

and

$$b_{i,j} = \begin{cases} 0, & \text{if } i < j; \\ \frac{f(N)}{f(i)} \prod_{\substack{k=j \\ k \neq i}}^{N} \frac{1}{1 - \frac{f(k)}{f(i)}}, & \text{if } i \geq j. \end{cases}$$

$$\mathbf{S}_{N|f}^{k+1} = \mathbf{D}_{N|f} \operatorname{diag}\left(f(1)^{k+1}, \ldots, f(N)^{k+1}\right) \left(\mathbf{D}_{N|f}\right)^{-1}$$

Remark.

$$a_{N,j} = 1$$

Example

Recall

$$f(x) = \frac{1}{x} \Rightarrow \mathbf{S}_{N|f} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

$$\begin{aligned} & \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} \\ = & N \cdot \left(\mathbf{S}_{N|f}^{k+1} \right)_{N,1} \\ = & N \cdot \left[\mathbf{D}_{N|f} \mathrm{diag} \left(f(1)^{k+1}, \ldots, f(N)^{k+1} \right) \left(\mathbf{D}_{N|f} \right)^{-1} \right]_{N,1} \end{aligned}$$

Example (Continued)

The last row of $\mathbf{D}_{N|f}$:

$$(1, \ldots, 1)$$

The first row of $\left(\mathbf{D}_{N|f}\right)^{-1}$:

$$\left(\frac{1}{N} \prod_{n=2}^{N} \frac{1}{1 - \frac{1}{n}}, \ldots, \frac{\ell}{N} \prod_{\substack{n=1 \\ n \neq \ell}}^{N} \frac{1}{1 - \frac{\ell}{n}}, \ldots, \prod_{n=1}^{N-1} \frac{1}{1 - \frac{N}{n}} \right).$$

$$\begin{aligned} \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} &= N \sum_{\ell=1}^{N} \frac{1}{\ell^{k+1}} \frac{\ell}{N} \prod_{\substack{n=1 \\ n \neq \ell}}^{N} \frac{1}{1 - \frac{\ell}{n}} \\ &= \sum_{j=1}^{N} (-1)^{\ell-1} \binom{N}{\ell} \frac{1}{\ell^k}. \end{aligned}$$

Another Example

Butler and Karasik obtained if $G\left(n,k\right)$ satisfies $G\left(n,n\right)=1$, $G\left(n,-k\right)=0$ and for $k\geq 1$

$$G\left(n,k\right)=G\left(n-1,k-1\right)+g(k)G\left(n-1,k\right),$$

then

$$G\left(N+k,N\right)=\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} g(n_1)\cdots g(n_k)$$

A note on nested sums, S. Butler and P. Karasik, *J. Integer Seq.* **13** (2010), Article 10.4.4.

Another Example

- When $k = 1$, an induction on N shows directly that

$$\sum_{N \geq n_1 \geq 1} g(n_1) = g(N) + G\left(N, N-1\right)$$
$$= g(N) G\left(N, N\right) + G\left(N, N-1\right)$$
$$= G\left(N+1, N\right).$$

- For inductive step in k, similarly from, we see

$$S\left(\underbrace{g, \ldots, g}_{k}, N, 1\right) = g(N)\left(\mathbf{S}_{N|g}^{k}\right)_{N,1} = g(N)\left(\mathbf{S}_{N|g}\left(\mathbf{S}_{N|g}^{k-1}\right)\right)_{N,1}$$
$$= \frac{1}{g(N)} \sum_{m=1}^{N} g(N) \cdot g(m) \cdot G\left(m+k-1, m\right)$$
$$= G\left(N+k, N\right),$$

by recurrence.

N=∞

Let $m = 1$, $N = \infty$, and $f_l(x) = 1/x^{s_l}$ for $l = 1, \ldots, k$. Then,

$$\mathcal{S}\left(f_1, \ldots, f_k; \infty, 1\right) := \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} = \zeta^*\left(s_1, \ldots, s_k\right),$$

and

$$\mathcal{A}\left(f_1, \ldots, f_k; \infty, 1\right) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} = \zeta\left(s_1, \ldots, s_k\right).$$

Example.

$$\zeta^*\left(2, 1\right) = 2\zeta(3).$$

$$\zeta^*(s_1, s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2),$$

$$\zeta^*(s_1, s_2, s_3) = \zeta(s_1, s_2, s_3) + \zeta(s_1 + s_2, s_3) + \zeta(s_1, s_2 + s_3) + \zeta(s_1 + s_2 + s_3).$$

Truncated & Generalized

$$\zeta^*(s_1, s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2),$$
$$\zeta^*(s_1, s_2, s_3) = \zeta(s_1, s_2, s_3) + \zeta(s_1 + s_2, s_3) + \zeta(s_1, s_2 + s_3) + \zeta(s_1 + s_2 + s_3).$$

Theorem. [L. Jiu and D. Shi]

$$\mathcal{S}(f, g; N-1, m) = \mathcal{A}(f, g; N, m) + \mathcal{A}(fg; N, m)$$

and

$$\begin{aligned}\mathcal{S}(f, g, h; N-1, m) = &\mathcal{A}(f, g, h; N, m) + \mathcal{A}(fg, h; N, m) \\ &+ \mathcal{A}(f, gh; N, m) + \mathcal{A}(fgh; N, m).\end{aligned}$$

KEY:

$$\mathbf{A}_{(N-1)|f} = (\delta_{i-1,j})_{N\times N}\,\mathbf{S}_{N|f}.$$

Open Question

Theorem. [M. Hoffman] For any real $i_1, \ldots, i_k > 1$,

$$\sum_{\sigma \in S_k} \zeta^* \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right) = \sum_{\text{partitions } \Pi \text{ of } \{1,\ldots,k\}} c(\Pi) \, \zeta(i_1, \ldots, i_k, \Pi),$$

where $\Pi = \{P_1, \ldots, P_\ell\}$ is a set partition of $\{1, \ldots, k\}$,

$$c(\Pi) := (|P_1| - 1)! \cdots (|P_\ell| - 1)!,$$

and

$$\zeta(i_1, \ldots, i_k, \Pi) = \prod_{s=1}^{\ell} \zeta\left(\sum_{j \in P_s} i_j\right).$$

And

$$\sum_{\sigma \in S_k} \zeta \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right) = \sum_{\text{partitions } \Pi \text{ of } \{1,\ldots,k\}} \tilde{c}(\Pi) \, \zeta(i_1, \ldots, i_k, \Pi),$$

where

$$\tilde{c}(\Pi) = (-1)^{k-\ell} c(\Pi).$$

$k \to \infty$

$$\lim_{k\to\infty} \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1\cdots n_k} = N.$$

Proof.

$$\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1\cdots n_k} = N \cdot \mathbb{P}\left(n_{k+1} = 1\right).$$

□