Generalized Stern polynomials: Their recursions and continued fractions

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Number Theory Seminar

Dalhousie University

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• Introduction

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(2) What is the Stern sequence,
(3) What does the sequence count?

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Moritz Abraham Stern at the University of Göttingen:
the first Jewish full professor at a German university.

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  as the sum of powers of 2,
  with no power used more than 2 times.

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- The Stern Diatomic Sequence can be created by recursion equations or by generating functions.
- Generalized Stern numbers count the number of representations of an integer n:
  as the sum of powers of any $b \geq 2$,
  with no power used more than b times.

Introduction

Stern sequence $\{a(n)\}_{n\geq 0}$ at $b = 2$ is defined by

$a(0) = 0,\ a(1) = 1$, and for $n \geq 1$,

$$\begin{aligned} a(2n) &= a(n), \\ a(2n+1) &= a(n) + a(n+1). \end{aligned}$$

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Sequence: $0, 1, 1, 2,$

$$\begin{aligned} &1, 3, 2, 3, \\ &1, 4, 3, 5, 2, 5, 3, 4, \\ &1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5, 1, \ldots \end{aligned}$$

Properties: $a(2^k)=1$, and *bimodal* between 1's.

• Hyperbinary expansions

Definition. A *hyperbinary expansion* (HBE) of an integer $n \geq 1$ is an expansion of n as a sum of powers of 2, each power being used at most 2 times.

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Example: The HBEs of $n = 12$ are

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&8+4,\\
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&4+4+2+2,\\
&4+4+2+1+1.
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The number of HBEs of an integer $n \geq 1$ is $a(n+1)$.

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Theorem (Reznick)

The number of HBEs of an integer $n \geq 1$ is $a(n+1)$.

Indeed, $a(12+1) = 5$.

Polynomial analogues

Two types of polynomials $a_{\ell,t}(n;z)$ at $\ell \in \{1,2\}$ for $t \geq 1$.

Recursions:

$$\begin{aligned} a_{1,t}(2n;z) &= z\, a_{1,t}(n;z^t), \\ a_{1,t}(2n+1;z) &= a_{1,t}(n;z^t) + a_{1,t}(n+1;z^t), \end{aligned}$$

$$\begin{aligned} a_{2,t}(2n;z) &= a_{2,t}(n;z^t), \\ a_{2,t}(2n+1;z) &= z\, a_{2,t}(n;z^t) + a_{2,t}(n+1;z^t). \end{aligned}$$

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Generating functions:

$$x \prod_{j\geq 0} \left(1 + z^{t^j} x^{2^j} + x^{2^{j+1}}\right) = \sum_{n\geq 0} a_{1,t}(n;z)\, x^n,$$

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• Originators - polynomials

Type 1 at $t = 1$

Sandi Klavžar

• Originators - polynomials

Type 1 at $t = 1$

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Type 2 at $t = 2$

Ken Stolarsky

n	$a_{1,t}(n;z)$	$a_{2,t}(n;z)$
1	1	1
2	z	1
3	$1+z^t$	$1+z$
4	z^{t+1}	1
5	$1+z^t+z^{t^2}$	$1+z+z^t$
6	$z+z^{t^2+1}$	$1+z^t$
7	$1+z^{t^2}+z^{t^2+t}$	$1+z+z^{t+1}$
8	z^{t^2+t+1}	1
9	$1+z^{t^2}+z^{t^2+t}+z^{t^3}$	$1+z+z^t+z^{t^2}$
10	$z+z^{t^2+1}+z^{t^3+1}$	$1+z^t+z^{t^2}$
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Proposition

For $t \geq 2$, polynomial coefficients are only 0 or 1,

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8	z^{t^2+t+1}	1
9	$1+z^{t^2}+z^{t^2+t}+z^{t^3}$	$1+z+z^t+z^{t^2}$
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Proposition

For $t \geq 2$, polynomial coefficients are only 0 or 1, and the exponent coefficients are also only 0 or 1.

Theorem

Let $\mathcal{P}_{n+1}$ be the set of exponents of z in expansion:

$$a_{2,t}(n+1; z) = \sum_{p \in \mathcal{P}_{n+1}} z^{p(t)}.$$

Then each HBE of n corresponds to exactly one polynomial in $\mathcal{P}_{n+1}$, as follows:

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then exactly the powers $2^{\alpha_1}, \ldots, 2^{\alpha_r}$ are repeated.

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Example. $a_{2,t}(n+1;z)$ at $n = 12$, $\mathcal{P}_{13} = \{1, t, 1+t^2, t+t^2\}$.

Last HBEs are characterized by repeated powers of 2:

- 2^0 and 2^2, so $12 = \mathbf{1} + \mathbf{1} + 2 + \mathbf{4} + \mathbf{4}$;
- 2^1 and 2^2, so $12 = \mathbf{2} + \mathbf{2} + \mathbf{4} + \mathbf{4}$.

Theorem

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Example. $a_{1,t}(n+1;z)$ at $n = 12$, $\mathcal{P}_{13} = \{t, t^3, t+t^3, t^2+t^3\}$.

Last HBEs are characterized by non-repeated powers of 2:

- 2^1 and 2^3, so $12 = 1 + 1 + \mathbf{2} + \mathbf{8}$;
- 2^3 and 2^2, so $12 = \mathbf{4} + \mathbf{8}$.

• Polynomial extension

Polynomials $\omega_{s,t}(n; y, z)$ in two variables y, z with $s, t \geq 1$.

Recursion:

$$\begin{aligned} \omega_{s,t}(2n; y, z) &= y\, \omega_{s,t}(n; y^s, z^t), \\ \omega_{s,t}(2n+1; y, z) &= z\, \omega_{s,t}(n; y^s, z^t) + \omega_{s,t}(n+1; y^s, z^t). \end{aligned}$$

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n	$\omega_{s,t}(n; y, z)$	n	$\omega_{s,t}(n; y, z)$
1	1	5	$y^s z + z^t + y^{s^2}$
2	y	6	$yz^t + y^{1+s^2}$
3	$z + y^s$	7	$z^{1+t} + y^{s^2} z + y^{s+s^2}$
4	y^{1+s}	8	y^{1+s+s^2}

• Explicit formula

Definition: Let $n = \sum_{j \geq 0} c_j\, 2^j$, then set $d_t(n) := \sum_{j \geq 0} c_j\, t^j$.

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Proposition

An explicit formula with $\binom{n}{k}^* \equiv \binom{n}{k} \pmod 2$ *becomes*

$$\omega_{s,t}(n+1; y, z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}^* y^{d_s(n-2k)} z^{d_t(k)}.$$

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Example: At $n = 5$, then $\omega_{s,t}(6; y, z) = y^{1+s^2} + yz^t$ as

k	$\binom{n-k}{k}$	$d_s(n-2k)$	$d_t(k)$	term
0	1	$1+s^2$	0	y^{1+s^2}
2	3	s^0	t^1	yz^t

Continued fractions

A continued fraction with limit $c = \frac{1}{2}(1+\sqrt{5})$ is

$$c := 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}.$$

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We use “K" notation as

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = b_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j}{b_j} = b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \dots$$

The *Rogers-Ramanujan continued fraction* in modified form is

$$R^{-1}(z) := 1 + \frac{z}{1+}\,\frac{z^2}{1+}\,\frac{z^3}{1+}\,\frac{z^4}{1+}\cdots$$

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We study continued fractions in extended forms at $t \geq 1$ like

$$c_t(z) := 1 + \frac{z^t}{1+}\,\frac{z^{t^2}}{1+}\,\frac{z^{t^3}}{1+}\,\frac{z^{t^4}}{1+}\cdots$$

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$$c_{p,t}(z) := p_0(z) + \frac{z^t}{p_1(z)+}\,\frac{z^{t^2}}{p_2(z)+}\,\frac{z^{t^3}}{p_3(z)+}\;\cdots$$

Continued fractions

Lehmer showed row maximums are F_n occurring at

$$\alpha_n := \frac{1}{3}\left(2^n - (-1)^n\right) \quad \text{and} \quad \beta_n := \frac{1}{3}\left(5 \cdot 2^{n-2} + (-1)^n\right).$$

n	1	2	3	4	5	6	7	8	9	10
α_n	1	1	3	5	11	21	43	85	171	341
β_n		2	3	7	13	27	53	107	213	427

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n	$a_{1,t}(\alpha_n; z)$	$a_{1,t}(\beta_n; z)$
2	1	z
3	$1 + z^t$	$1 + z^t$
4	$1 + z^t + z^{t^2}$	$1 + z^{t^2} + z^{t^2+t}$
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Polynomials have recurrences like Fibonacci numbers.

Definition

For a fixed integer $t \geq 1$ we have

$$a_{1,t}(\alpha_{n+1}; z) = a_{1,t}(\alpha_n; z^t) + z^t a_{1,t}(\alpha_{n-1}; z^{t^2}),$$
$$a_{1,t}(\beta_{n+1}; z) = a_{1,t}(\beta_n; z^t) + z^t a_{1,t}(\beta_{n-1}; z^{t^2}).$$

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Proposition

For integers $t \geq 1$ and $n \geq 3$ we have

$$\frac{a_{1,t}(\alpha_{n+1}; z)}{a_{1,t}(\alpha_n; z^t)} = 1 + \cfrac{z^t}{1 + \cfrac{z^{t^2}}{1 + \cfrac{z^{t^3}}{1 + \cdots z^{t^{n-1}}}}}$$

$$= 1 + \frac{z^t}{1+}\,\frac{z^{t^2}}{1+}\,\frac{z^{t^3}}{1+} \cdots \frac{z^{t^{n-2}}}{1+}\,\frac{z^{t^{n-1}}}{1}.$$

Proposition

There is a unique function, analytic for $|z| < 1$, defined by

$$F_{1,t}(z) := \lim_{n\to\infty} a_{1,t}(\alpha_n; z) = \lim_{n\to\infty} a_{1,t}(\beta_n; z)$$
$$= 1 + z^t + z^{t^2} + z^{t^3} + z^{t^3+t} + z^{t^4} + z^{t^4+t} + z^{t^4+t^2} + \ldots$$

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Proposition

For every integer $t \geq 2$ we have for $z \in \mathbb{C}$ with $|z| < 1$,

$$\frac{F_{1,t}(z)}{F_{1,t}(z^t)} = 1 + \frac{z^t}{1+} \frac{z^{t^2}}{1+} \frac{z^{t^3}}{1+} \frac{z^{t^4}}{1+} \dots$$

When $t = 1$ we get $1 + \frac{z}{1+} \frac{z}{1+} \frac{z}{1+} \frac{z}{1+} \cdots = (1 + \sqrt{1 + 4z})/2$.

When $z = 1$ we get $1 + \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \cdots = (1 + \sqrt{5})/2 = \Phi$.

Continued fractions by index runs

Graham, Knuth, Patashnik consider when positive integer n has the binary representation

$$n = (1^{r_1} 0^{r_2} \cdots 1^{r_\ell})_2\,, \quad r_j > 0, \quad 2^k \leq n < 2^{k+1},$$

where 1^{r_1} indicates the binary digit 1 is repeated r_1-times.

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Theorem (GKP)

Stern number $a(n)$ is the numerator of the continued fraction

$$r_1 + \cfrac{1}{r_2 + \cfrac{1}{\ddots + \cfrac{1}{r_\ell}}}.$$

• Originators - continued fractions

$a_{1,s}(n;y)$ at $s=1$

Andrzej Schinzel

• Originators - continued fractions

$a_{1,s}(n; y)$ at $s = 1$

Andrzej Schinzel

$\omega_{s,t}(n; y, z)$ at $s, t = 1$

Toufik Mansour

Continued fractions

Mansour extended the Stern number case to polynomials $\omega_{s,t}(n; y, z)$ at $s = t = 1$.

Theorem (Mansour)

Polynomial $\omega_{1,1}(n; y, z)$ *is the numerator of continued fraction*

$$[r_1]_{y,z} + \cfrac{y^{r_1}}{z[r_2]_{y,1} + \cfrac{y^{r_2} z^{r_3}}{[r_3]_{y,z} + \cfrac{y^{r_3}}{z[r_4]_{y,1} + \cfrac{y^{r_4} z^{r_5}}{\ddots \over [r_\ell]_{y,z}}}}}, \quad [r]_{y,z} := \frac{y^r - z^r}{y - z}.$$

We extend numerators to polynomials $\omega_{s,t}(n; y, z)$ for $s, t \geq 1$ and identify denominators as polynomials $\overline{\omega}_{s,t}(n; y, z)$.

Definition

At fixed integers $s, t \geq 1$, we define polynomials $\overline{\omega}_{s,t}(n; y, z)$ by

$$\overline{\omega}_{s,t}(2n; y, z) = y\,\overline{\omega}_{s,t}(n; y^s, z^t),$$

$$\overline{\omega}_{s,t}(2n+1; y, z) = \begin{cases} \overline{\omega}_{s,t}(n; y^s, z^t) + z\,\overline{\omega}_{s,t}(n+1; y^s, z^t), & n \neq 2^\nu, \\ z^{t^{\nu+1}}\overline{\omega}_{s,t}(n; y^s, z^t) + z\,\overline{\omega}_{s,t}(n+1; y^s, z^t), & n = 2^\nu, \end{cases}$$

where $\nu \geq 0$ is some integer.

Theorem

We obtain the quotient of polynomials for $2^k \le n < 2^{k+1}$ *as*

$$\frac{\omega_{s,t}(n; y, z)}{\overline{\omega}_{s,t}(2^{k+1} - n; y, z)}$$

with the required continued fraction expansion.

Theorem

We obtain the quotient of polynomials for $2^k \le n < 2^{k+1}$ as

$$\frac{\omega_{s,t}(n; y, z)}{\overline{\omega}_{s,t}(2^{k+1} - n; y, z)}$$

with the required continued fraction expansion.

Example 4. Let $n = 27 = (11011)_2$, so $r_1 = 2$, $r_2 = 1$, $r_3 = 2$. Then the continued fraction is

$$(y^{s^4} + z^{t^3}) + \cfrac{y^{s^3+s^4}}{z^{t^2} + \cfrac{y^{s^2} z^{1+t}}{y^s + z}}.$$

Expanding this, we get numerator polynomial as $\omega_{s,t}(27; y, z)$ and denominator with index $32 - 27 = 5$ as

$$\overline{\omega}_{s,t}(5; y, z) \;=\; y^{s^2} z^{1+t} + y^s z^{t^2} + z^{1+t^2}.$$

Corollary

At $n = (1^r 0^r \cdots 1^r)_2$ *for fixed* r *and* $\omega_{s,t}(n; y, 1) = a_{1,s}(n; y)$, *then*

$$\lim_{n\to\infty} \frac{\omega_{s,t}(\alpha_{n+1}(r); y, 1)}{\overline{\omega}_{s,t}(\alpha_n(r); y, 1)} = a_{1,s}(2^r - 1; y) + \operatorname*{K}_{j=1}^{\infty} \frac{a_{1,s}(2^r; y^{s^{jr}})}{a_{1,s}(2^r - 1; y^{s^{jr}})}.$$

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Example. At $r = 2$ with $a_{1,s}(3; y) = 1 + y^s$, $a_{1,s}(4; y) = y^{1+s}$, then we get

$$1 + y^s + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{y^{s^{2j} + s^{2j+1}}}{1 + y^{s^{2j+1}}},$$

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which at $y = 1$ becomes

$$2 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{2} = 2.414\ldots = 1 + \sqrt{2},$$

as the limit ratio of Pell numbers.

• Extend to Lucas sequence

Recall: If

$$\alpha_n := \frac{1}{3}\left(2^n - (-1)^n\right) \quad \text{and} \quad \beta_n := \frac{1}{3}\left(5 \cdot 2^{n-2} + (-1)^n\right)$$

Then

$$a(\alpha_n) = a(\beta_n) = F_n \qquad (n \geq 2),$$

where $\{a(m)\}$ is Stern's diatomic sequence (Lehmer, 1929).

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This is a special case of a more general relation:

For a fixed $k \in \mathbb{N}$ define the *Lucas function* $U_n(k) = U_n(k, -1)$ by $U_0(k) = 0$, $U_1(k) = 1$, and

$$U_{n+1}(k) = k\,U_n(k) + U_{n-1}(k) \qquad (n \geq 1).$$

Now define, again for a fixed $k \in \mathbb{N}$,

$$\alpha_n(k) := \frac{2^{nk} - (-1)^n}{2^k + 1} \qquad (n \geq 0),$$

$$\beta_n(k) := \frac{(2^k + 3)2^{nk-k-1} + (-1)^n}{2^k + 1} \qquad (n \geq 2).$$

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Proposition

For all $k \geq 1$ and $n \geq 2$,

$$a(\alpha_n(k)) = a(\beta_n(k)) = U_n(k).$$

Recall: The recursion at $k = 1$:

$$a_{1,t}(\alpha_{n+1}; z) = a_{1,t}(\alpha_n; z^t) + z^t a_{1,t}(\alpha_{n-1}; z^{t^2}) \tag{1}$$

was used to obtain one of our continued fractions.

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Generalization:

Lemma

With fixed $t, k \in \mathbb{N}$ we have for all $n \geq 1$,

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Note:

$$\begin{aligned} a_{1,t}(2^k - 1; z) &= 1 + z^{t+\cdots+t^{k-1}} + z^{t^2+\cdots+t^{k-1}} + \cdots + z^{t^{k-1}}, \\ a_{1,t}(2^k; z) &= z^{1+t+\cdots+t^{k-1}}. \end{aligned}$$

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Therefore (1) is indeed a special case of (2).

As before, the Lemma leads to a finite continued fraction (for simplicity, write $a(m; z)$ for $a_{1,t}(m; z)$):

Proposition

$$\frac{a(\alpha_{n+1}(k); z)}{a(\alpha_n(k); z^{t^k})} = a(2^k - 1; z) + \frac{a(2^k; z^{t^k})}{a(2^k - 1; z^{t^k})+}\ \frac{a(2^k; z^{t^{2k}})}{a(2^k - 1; z^{t^{2k}})+} \cdots$$
$$\cdots \frac{a(2^k; z^{t^{(n-2)k}})}{a(2^k - 1; z^{t^{(n-2)k}})+}\ \frac{a(2^k; z^{t^{(n-1)k}})}{a(2^k - 1; z^{t^{(n-1)k}})}.$$

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Example $k = 1$**.** For $a(2^k - 1; z) = 1$, $a(2^k; z) = z$, then

$$\frac{a_{1,t}(\alpha_{n+1}; z)}{a_{1,t}(\alpha_n; z^t)} = 1 + \frac{z^t}{1+} \, \frac{z^{t^2}}{1+} \, \frac{z^{t^3}}{1+} \cdots \frac{z^{t^{n-2}}}{1+} \, \frac{z^{t^{n-1}}}{1}.$$

Example $k = 2$**.** For $a(2^k - 1; z) = 1 + z^t$, $a(2^k; z) = z^{1+t}$, then

$$\frac{a_{1,t}(\alpha_{n+1}(2); z)}{a_{1,t}(\alpha_n(2); z^{t^2})} = (1 + z^t) + \frac{z^{t^2+t^3}}{(1 + z^{t^3})+} \frac{z^{t^4+t^5}}{(1 + z^{t^5})+} \cdots$$

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• All this can also be done with $a_{1,t}(\beta_n(k); z)$ and for the type-2 case; we get different continued fractions of a similar nature.

• Ternary case ($b=3$)

Polynomials $\omega_T(n;Z)$ have $Z=(x,y,z)$, $T=(r,s,t)$.

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Recursion: With $\omega(0; Z) = 0$, $\omega(1; Z) = 1$, then for $n \geq 1$:

$$\begin{aligned}
\omega(3n-1; x, y, z) &= x\, \omega(n; x^r, y^s, z^t) \\
\omega(3n+0; x, y, z) &= y\, \omega(n; x^r, y^s, z^t) \\
\omega(3n+1; x, y, z) &= z\, \omega(n; x^r, y^s, z^t) + \omega(n+1; x^r, y^s, z^t).
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\end{aligned}$$

n	$\omega_T(n; Z)$	n	$\omega_T(n; Z)$
2	x	6	$x^r y$
3	y	7	$y^s + x^r z$
4	$x^r + z$	8	$x\, y^s$
5	x^{1+r}	9	y^{1+s}

Stern b-ary extension

Any integer base $b \geq 2$, polynomials $\omega_T^b(n; Z)$ have $Z = (z_1, \ldots, z_b)$, $T = (t_1, \ldots, t_b)$.

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$$\zeta \prod_{j=0}^{\infty} \left(1 + z_1^{t_1^j} \zeta^{b^j} + z_2^{t_2^j} \zeta^{2 \cdot b^j} + \cdots + z_b^{t_b^j} \zeta^{b \cdot b^j} \right) = \sum_{n=0}^{\infty} \omega_T(n; Z)\, \zeta^n.$$

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Recursions: With $\omega_T(0; Z) = 0, \omega_T(1; Z) = 1$, then for $n \geq 1$:

$$\begin{aligned} \omega_T(b(n-1) + j + 1; Z) &= z_j \, \omega_T(n; Z^T) \quad (1 \leq j \leq b-1), \\ \omega_T(b\,n + 1; Z) &= z_b \, \omega_T(n; Z^T) + \omega_T(n+1; Z^T). \end{aligned}$$

• Stern b-ary extension

Polynomials encode each hyper b-ary expansion h in $\mathbb{H}_{b,n}$ as

$$\omega_T(n+1; z_1, \ldots, z_b) = \sum_{h \in \mathbb{H}_{b,n}} z_1^{p_{h,1}(t_1)} \cdots z_b^{p_{h,b}(t_b)}.$$

If we write exponent polynomials as

$$p_{h,j}(t_j) = t_j^{\tau_j(1)} + t_j^{\tau_j(2)} + \cdots + t_j^{\tau_j(\nu_j)},$$

the powers used exactly j times in the representation of n are

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Example: For $b = 3$, $n = 36$, then

$$\omega_T(37; x, y, z) = x^{r^2+r^3} + x^{r^3} y^s z^1 + x^{r^3} z^t + y^s z^{1+t^2} + z^{t+t^2}.$$

$h \in \mathbb{H}_{3,36}$	$p_{h,1}(r)$	$p_{h,2}(s)$	$p_{h,3}(t)$
$3^3 + 3 + 3 + 1 + 1 + 1$	r^3	s^1	t^0
$3^3 + 3 + 3 + 3$	r^3	0	t^1

• Stern b-ary maxima

If $b \geq 2$, there are $2(b-1)$ locations of Fibonacci numbers F_n being maxima m in row n with $b^{n-2} \leq m \leq b^{n-1}$.

At $1 \leq j \leq b-1$, those indices α_n^b, β_n^b are given in base b:

$$\alpha_{n,j}^b = \begin{cases} (j\,0(10)^{\ell-2}1)_b & \text{if } \; n = 2\ell, \\ (j\,0(10)^{\ell-2}11)_b & \text{if } \; n = 2\ell+1. \end{cases}$$

$$\beta_{n,j}^b = \begin{cases} (j\,(10)^{\ell-2}11)_b & \text{if } \; n \text{ is odd}, \\ (j\,(10)^{\ell-1}1)_b & \text{if } \; n \text{ is even}. \end{cases}$$

• Lucas sequences

Recall the Lucas function at $U_0(k) = 0, U_1(k) = 1$,

$$U_{n+1}(k) = k\, U_n(k) + U_{n-1}(k) \quad (n \leq 1).$$

For $b \geq 2$, Stern numbers $a(n)$ at indices $\alpha^b_{n,j}(k), \beta^b_{n,j}(k)$ are

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$$\alpha^b_{n,j}(k) = \begin{cases} (j^1 1^{k-1} 0^k 1^k 0^k \cdots 1^k 0^{k-1} 1^1)_b & \text{if } \; n \text{ is odd,} \\ (j^1 1^{k-1} 0^k 1^k 0^k \cdots 1^k 0^k 1^k)_b & \text{if } \; n \text{ is even.} \end{cases}$$

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Recursions for $b = 3$

Recursions for $b = 3$ with $Z = (x, y, z),\ T = (r, s, t)$ at $\alpha^b_{n,j}(k)$:

At $k = 1$, polynomial recursions with $\alpha^b_{2n,j} = \alpha^b_{2n,j}(k)$ are

$$\begin{aligned}\omega_T(\alpha^b_{2n,j}; Z) &= 1 \cdot \omega_T(\alpha^b_{2n-1,j}; Z^T) + y^s z \cdot \omega_T(\alpha^b_{2n-2,j}; Z^{T^2}),\\ \omega_T(\alpha^b_{2n+1,j}; Z) &= z \cdot \omega_T(\alpha^b_{2n,j}; Z^T) \quad + \quad x^r \cdot \omega_T(\alpha^b_{2n-1,j}; Z^{T^2}).\end{aligned}$$

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At $k = 2$, polynomial recursions with $\alpha^b_{2n,j} = \alpha^b_{2n,j}(k)$ are

$$\begin{aligned}\omega_T(\alpha^b_{2n,j}; Z) &= (y^s z + z^t) \cdot \omega_T(\alpha^b_{2n-1,j}; Z^{T^k})\\ &\quad + x^{r^2(1+r)} \cdot \omega_T(\alpha^b_{2n-2,j}; Z^{T^{2k}}),\end{aligned}$$

$$\begin{aligned}\omega_T(\alpha^b_{2n+1,j}; Z) &= (x^r + z) \cdot \omega_T(\alpha^b_{2n,j}; Z^{T^k})\\ &\quad + y^{s^2(1+s)} z^{1+t} \cdot \omega_T(\alpha^b_{2n-1,j}; Z^{T^{2k}}).\end{aligned}$$

• Recursions consolidated for $b \geq 2$:

Definition: For an integer $b \geq 2$ we have sets with cardinality b for parameters T, S and variables Z, Y given by

$$T = (t_1, t_2, \ldots, t_{b-1}, t_b), \qquad S = (s, s, \ldots, s, t),$$
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Proposition

Polynomial recursions with $\alpha^b_{2n,j} = \alpha^b_{2n,j}(k)$ *are*

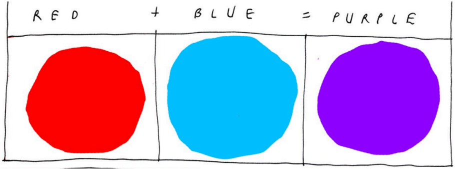
$$\begin{aligned}\omega_T(\alpha^b_{2n+2,j}; Z) &= z\,\omega_S(\tfrac{b^k-1}{b-1}; Y) \cdot \omega_T(\alpha^b_{2n+1,j}; Z^{T^k}) \\ &\quad + x^{r^k(\frac{r^k-1}{r-1})} \cdot \omega_T(\alpha^b_{2n,j}; Z^{T^{2k}}), \\ \omega_T(\alpha^b_{2n+1,j}; Z) &= \omega_T(\tfrac{b^k-1}{b-1}; Z) \cdot \omega_T(\alpha^b_{2n,j}; Z^{T^k}) \\ &\quad + y^{s^k(\frac{s^k-1}{s-1})} z^{\frac{t^k-1}{t-1}} \cdot \omega_T(\alpha^b_{2n-1,j}; Z^{T^{2k}}).\end{aligned}$$

Example: At $b = 3$, $k = 1$, and $\omega_T^3(\frac{b^k-1}{b-1}; Z) = 1$, then

$$\lim_{n\to\infty} \frac{\omega_T(\alpha^3_{2n+1}; Z)}{\omega_T(\alpha^3_{2n}; Z^T)} = 1 + \operatorname*{K}_{j=0}^{\infty} \frac{y^{s^{2j+1}} z^{t^{2j}}}{z^{t^{2j}}+} \, \frac{x^{r^{2j+1}}}{1},$$

$$\lim_{n\to\infty} \frac{\omega_T(\alpha^3_{2n+2}; Z)}{\omega_T(\alpha^3_{2n+1}; Z^T)} = z + \operatorname*{K}_{j=0}^{\infty} \frac{x^{r^{2j+1}}}{1+} \, \frac{y^{s^{2j+1}} z^{t^{2j}}}{z^{t^{2j}}}.$$

Thank you



Thank you