The goal of this talk is to give a method for computing a certain invariant of subsets of metric spaces which is of interest to analysts, and which has its origins in physics. To get there we will need some ideas from algebra and number theory, but I would like to start with some notation which is probably familiar to everyone, but which will avoid some misunderstanding later if I spell it out now:

Let p be a prime (which will be 2 in most of my examples)

and, for $u, v \in \mathbb{Z}$ or $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$; let

$$v_p(u) = \text{the number of times } p \text{ divides } z$$
$$= \text{the } p\text{-adic valuation of } z$$

and $\rho_p(u,v) = p^{-v_p(u-v)}$ = the p-adic distance between z and w.

$\rho_p(z,w)$ is in fact a metric of a special kind called either an ultrametric or a non-archemedean metric

Since it satisfies the strong triangle inequality

$$\rho_p(u,v) \leq \max(\rho_p(u,w), \rho_p(v,w))$$

with the equality unless $\rho(u,w) = \rho(v,w)$.

(A point which will be significant later is that because of the minus sign in the formula for ρ_p minimizing wrt v_p corresponds to maximizing wrt ρ_p)

1/ Integer Valued Polynomials:

If $S = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$ or $\hat{\mathbb{Z}}_p$ Then

$$\mathrm{Int}(S) = \{ f(x) \in \mathbb{Q}[x] \mid f(S) \subseteq \mathbb{Z} \text{ or } \mathbb{Z}_{(p)} \text{ or } \hat{\mathbb{Z}}_p \}$$

and a question of interest is what these polynomials look like. The classic example is $S = \mathbb{Z}$ in which case the binomial polynomials $\binom{x}{n} = \prod_{i=0}^{n-1} \frac{x-i}{n-i}$ form a basis over $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$ or $\hat{\mathbb{Z}}_p$

For other S Manjul Bhargava showed that a similar basis could be constructed, at least for $\mathbb{Z}_{(p)}$ or $\hat{\mathbb{Z}}_p$ by the polynomials $\prod_{i=0}^{n-1} \frac{(x-a_i)}{(a_n-a_i)}$

where the a_i's are picked to be what is called a p-ordering of S: a_0 is any element of S

a_1 minimizes $v_p(a_1-a_0)$ over $a_1 \in S$

a_2 minimizes $v_p((a_2-a_1)(a_2-a_0))$ over $a_2 \in S$

etc.

An interesting property of this construction, which is not obvious, is that the numbers

$\alpha_S(n) = v_p\left(\prod_{i=0}^{n-1} (a_n - a_i)\right)$ are independent of how the a_i's are picked, but depend only on S, and that the sequences $\left\{ \frac{\alpha_S(n)}{n} \right\}$ is convergent, For example for $S = \mathbb{Z}$, $a_i = i$ is such a p-ordering, so that

$$\alpha_{\mathbb{Z}}(n) = \nu_p(n!) = \frac{n - \sum n_i}{p-1} \quad \text{if } n = \sum n_i p^i$$

and the limit $\lim \frac{\alpha_{\mathbb{Z}}(n)}{n} = \lim \frac{1 - (\sum n_i / n)}{p-1} = \frac{1}{p-1}$

This limit is called the valuative capacity of S, $\omega(S)$.

Some other examples are:

$$S = \mathbb{Z} \setminus p\mathbb{Z} = \text{"integer prime to } p\text{"}$$
$$\omega(S) = p/(p-1)^2$$

$p = 2$, $S = \mathbb{Z} \setminus 4\mathbb{Z}$ $\quad \omega(S) = 2/3$.

Bhargava showed that the p-orderings actually have a stronger minimization property:

If $\{a_i\}$ is a p-ordering of S then for each n $\{a_i\}_0^{n-1}$ minimizes

$$\nu_p\left(\prod_{0 \le i < j < n} (a_i - a_j)\right)$$

over all n-tuples in S. Another way of saying this is that $\{a_i\}_0^{n-1}$ maximizes

$$\prod_{0 \le i < j < n} \rho_p(a_i, a_j).$$

This connects p-orderings to an idea introduced in 1923 by Michael Fekete. If C is any compact subset of $\mathbb{C}$, then a Fekete n-tuple is a set of n points $\{a_i\}_{i=0}^{n-1}$ such that

$$\prod_{0 \le i < j < n} \|a_i - a_j\| \text{ is maximal.}$$

Physically this corresponds to making C out of a conducting substance, such as copper, introducing n electrons, and seeing how they arrange themselves in C. Fekete showed that for such sets the limit

$$\lim_{n \to \infty} \prod_{0 \le i < j < n} \|a_i - a_j\|^{\frac{2}{n(n-1)}}$$

always existed, and called it the transfinite diameter of C. Of course there is nothing special about $\mathbb{C}$ with its usual metric here. The definitions carry over to any metric space and compact subset.

Bhargava's lemma shows that for any n the first n terms of a p-ordering are a Fekete set for $S \subseteq \mathbb{Z}_p$.

The definition of p-sequence can be extended to any compact subset of a metric space: Given $a_0 \ldots a_{n-1}$ pick a_n to maximize $\prod \rho(a_n, a_i)$

A natural question is whether this always gives Fekete n-tuples, but the answer is no in general. For example in $\mathbb{R}$ or $\mathbb{C}$ with the usual metric. A refined question is whether it works in an ultra metric space.

This would require having an extension of the result that the sequence $\{\prod_{i=0}^{n} \rho_1(a_n, a_i)\}_n$ is independent of the choice of the a_n's, and Bhargava's proof of this in $\mathbb{Z}$ makes essential use of the arithmetic structure of $\mathbb{Z}$.

Lets return for a moment to how you compute p-sequences and ω in $\mathbb{Z}_{(p)}$. There are 3 useful results

1/ translation invariance $\qquad \omega(S+a) = \omega(S)$

2/ scaling $\qquad \omega(p\cdot S) = \omega(S) + 1$

3/ decomposition: if $S = A \amalg B$ and $\rho_p(a,b) = 1$ for $a \in A, b \in B$ then a p-ordering of S is the shuffle of ones for A and B and

$$\omega(S) = \frac{1}{\frac{1}{\omega(A)} + \frac{1}{\omega(B)}}$$

eg $S = \mathbb{Z} \setminus p\mathbb{Z} = 1 + p\mathbb{Z} \amalg 2 + p\mathbb{Z} \amalg \cdots \amalg (p-1) + p\mathbb{Z}$ $\qquad \omega(i+p\mathbb{Z}) = \omega(p\mathbb{Z}) = \omega(\mathbb{Z}) + 1 = \frac{1}{p-1} + 1 = \frac{p}{p-1}$

$$\omega(S) = \frac{1}{\frac{1}{\left(\frac{p}{p-1}\right)} + \cdots + \frac{1}{\frac{p}{(p-1)}}} = \frac{1}{\frac{(p-1)^2}{p}} = \frac{p}{(p-1)^2}$$

Note that 3/ does not make use of the arithmetic structure of $\mathbb{Z}$ and its proof carries over to any ultra metric space.

Using this one can prove, by induction on the number of terms in the sequence, that

1/ If S is a compact subset of an ultrametric space then the sequence $\left\{ \prod_{0}^{n-1} \rho(a_n, a_i) \right\}_n$ is independent of the choice of a_n's in a p-ordering of S.

2/ If S is a compact subset of an ultrametric space then the first n terms of a p-ordering give a Fekete n-tuple

3/ If $\mu(S) = \lim \left(\prod_{0}^{n-1} \rho(a_n, a_i) \right)^{1/n}$

Then $\mu(S)$ equals the transfinite diameter of S and that μ satisfies the formula

$$\log\left(\frac{\mu(S)}{d}\right) = \frac{1}{\dfrac{1}{\log\frac{\mu(A)}{d}} + \dfrac{1}{\log\left(\frac{\mu(B)}{d}\right)}}$$

If $S = A \amalg B$ and $\rho(a,b) = d$ for all $a \in A$, $b \in B$.

Transfinite diameter, outer diameter, capacity, chebychev constant, robin's constant,