A Composite Problem

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Overview

The Ring of Integer-Valued Polynomials

The set

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

of rational polynomials taking integer values over the integers forms a subring of $\mathbb{Q}[x]$ called the *ring of integer-valued polynomials* (IVPs).

$\mathrm{Int}(\mathbb{Z})$ is a polynomial ring and has basis $\left\{\binom{x}{k} : k \in \mathbb{Z}_{>0}\right\}$ as a $\mathbb{Z}$-module, with

$$\binom{x}{k} := \frac{x(x-1)\cdots(x-(k-1))}{k!}\ , \qquad \binom{x}{0} = 1\ , \qquad \binom{x}{1} = x\ .$$

This basis is a *regular basis*, meaning that the basis contains exactly one polynomial of degree k for $k \geq 1$.

p-orderings

The study of IVPs on subsets of the integers greatly benefited from the introduction of p-orderings by Bhargava [1].

Definition

Let S be a subset of $\mathbb{Z}$ and p be a fixed prime. A *p-ordering of S* is a sequence $\{a_i\}_{i=0}^{\infty} \subseteq S$ defined as follows: choose an element $a_0 \in S$ arbitrarily. Further elements are defined inductively where, given $a_0, a_1, \ldots, a_{k-1}$, the element $a_k \in S$ is chosen so as to minimize the highest power of p dividing

$$\prod_{i=0}^{k-1} (a_k - a_i) .$$

p-sequences

The choice of a p-ordering gives a corresponding sequence:

Definition

The *associated p-sequence of S*, denoted $\{\alpha_{S,p}(k)\}_{k=0}^{\infty}$, is the sequence wherein the k^{th} term $\alpha_{S,p}(k)$ is the power of p minimized at the k^{th} step of the process defining a p-ordering. More explicitly, given a p-ordering $\{a_i\}_{i=0}^{\infty}$ of S,

$$\alpha_{S,p}(k) = \nu_p\left(\prod_{i=0}^{k-1}(a_k - a_i)\right) = \sum_{i=0}^{k-1} \nu_p(a_k - a_i) .$$

Though the choice of a *p*-ordering of S is not unique, the associated *p*-sequence of a subset $S \subseteq \mathbb{Z}$ is independent of the choice of *p*-ordering [1].

These *p*-orderings can be used to define a generalization of the binomial polynomials to a specific set $S \subseteq \mathbb{Z}$ which serve as a basis for the integer-valued polynomials of S over $\mathbb{Z}$,

$$\mathrm{Int}(S, \mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(S) \subseteq \mathbb{Z}\} .$$

IVPs over Matrix Rings

We are particularly interested in studying IVPs over matrix rings.

We denote the set of rational polynomials mapping integer matrices to integer matrices by

$$\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z})) = \{f \in \mathbb{Q}[x] : f(M) \in M_n(\mathbb{Z}) \text{ for all } M \in M_n(\mathbb{Z})\}\ .$$

We know from Cahen and Chabert [2] that $\mathrm{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ has a regular basis, but it is not easy to describe using a formula in closed form [3].

Link to Maximal Orders

Finding a regular basis for $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z}))$ is related to finding a regular basis for its integral closure, and we understand the latter object through studying its localizations at rational primes.

If p is a fixed prime, D is a division algebra of degree n^2 over $K = \mathbb{Q}_p$, and Δ_n is its maximal order, then we obtain the following useful result:

Proposition ([3], 2.1)

The integral closure of $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z})_{(p)})$ is $\text{Int}_{\mathbb{Q}}(\Delta_n)$.

Thus, the problem of describing the integral closure of $\text{Int}_{\mathbb{Q}}(M_n(\mathbb{Z})_{(p)})$ is exactly that of describing $\text{Int}_{\mathbb{Q}}(\Delta_n)$, and so we move our attention towards studying IVPs over maximal orders.

An Analogue to p-orderings

Definition-Proposition ([4], 1.1, 1.2)

Let K be a local field with valuation ν, D a division algebra over K to which ν extends, Δ the maximal order in D, and S a subset of Δ.

- A *ν-ordering* of S is a sequence $\{a_i\} \subseteq S$ such that for each $k > 0$, the element a_k minimizes the quantity $\nu(f_k(a_0, \ldots, a_{k-1})(a))$ over $a \in S$, where $f_k(a_0, \ldots, a_{k-1}(x))$ is the minimal polynomial of the set $\{a_0, a_1, \ldots, a_{k-1}\}$, with the convention that $f_0 = 1$. We call $\alpha_S = \{\alpha_S(k) = \nu(f_k(a_0, \ldots, a_{k-1})(a_k)) : k = 0, 1, \ldots\}$ the *ν-sequence* of S.
- Additionally, let $\pi \in \Delta$ be a uniformizing element. Then the ν-sequence α_S depends only on the set S, and not on the choice of ν-ordering. The sequence of polynomials

$$\{\pi^{-\alpha_S(k)} f_k(a_0, \ldots, a_{k-1})(x) : k = 0, 1, \ldots\}$$

forms a regular Δ-basis for the Δ-algebra of polynomials which are integer-valued on S.

In order to use this proposition, we need to be able to construct a ν-ordering for the maximal order Δ_n. A recursive method for constructing ν-orderings for elements of a maximal order is based on two lemmas.

Lemma (see [4], 6.2)

Let $\{a_i : i = 0, 1, 2, \dots\}$ be a ν-ordering of a subset S of Δ_n with associated ν-sequence $\{\alpha_S(i) : i = 0, 1, 2, \dots\}$ and let b be an element in the centre of Δ_n. Then:

i) $\{a_i + b : i = 0, 1, 2, \dots\}$ is a ν-ordering of $S + b$, and the ν-sequence of $S + b$ is the same as that of S

ii) If p is the characteristic of the residue field of K (so that $(p) = (\pi)^n$ in Δ_n), then $\{pa_i : i = 0, 1, 2, \dots\}$ is a ν-ordering for pS and the ν-sequence of pS is $\{\alpha_S(i) + in : i = 0, 1, 2, \dots\}$

Lemma ([4], 5.2)

Let S_1 and S_2 be disjoint subsets of S with the property that there is a non-negative integer k such that $\nu(s_1 - s_2) = k$ for any $s_1 \in S_1$ and $s_2 \in S_2$, and that S_1 and S_2 are each closed with respect to conjugation by elements of Δ_n. If $\{b_i\}$ and $\{c_i\}$ are ν-orderings of S_1 and S_2 respectively with associated ν-sequence $\{\alpha_{S_1}(i)\}$ and $\{\alpha_{S_2}(i)\}$, then the ν-sequence of $S_1 \cup S_2$ is the sum of the linear sequence $\{ki : i = 0, 1, 2, \dots\}$ with the shuffle $\{\alpha_{S_1}(i) - ki\} \wedge \{\alpha_{S_2}(i) - ki\}$, and this shuffle applied to $\{b_i\}$ and $\{c_i\}$ gives a ν-ordering of $S_1 \cup S_2$.

The theory presented in the previous slides is utilized by Evrard and Johnson [3] to construct a ν-order for Δ_2 and establish a ν-sequence and regular basis for the IVPs on Δ_2 when the division algebra D is over the local field $\mathbb{Q}_2$.

We would like to extend these results to the general case, in order to find a regular basis for the integer-valued polynomials on Δ_n over the local field $\mathbb{Q}_2$.

Constructing Δ_n

We can use these lemmas by decomposing Δ_n as a union of subsets to which the lemmas apply. Let $\mathbb{Q}_2$ denote the 2-adic numbers, and let ζ be a $(2^n - 1)^{\text{th}}$ root of unity. Let θ be the automorphism of $\mathbb{Q}_2(\zeta)$ that maps $\theta(\zeta) = \zeta^2$. Define $n \times n$ matrices ω_n and π_n as:

$$\omega_n = \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & \theta(\zeta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta^{n-1}(\zeta) \end{pmatrix} \quad \pi_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 2 & 0 & \cdots & 0 \end{pmatrix}$$

The maximal order Δ_n with which we concern ourselves is

$$\Delta_n = \mathbb{Z}_2[\omega_n, \pi_n]$$

where $\mathbb{Z}_2$ denotes the 2-adic integers.

$$\Delta_n = \mathbb{Z}_2[\omega_n, \pi_n]$$

$$\omega_n = \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & \theta(\zeta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta^{n-1}(\zeta) \end{pmatrix} \quad \pi_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 2 & 0 & \cdots & 0 \end{pmatrix}$$

The elements ω_n and π_n observe the commutativity relation $\pi_n\omega_n = \omega_n^2\pi_n$, and note also that $\pi_n^n = 2I_n$. An element $z \in \Delta_n$ can be expressed as a $\mathbb{Z}_2$-linear combination of the elements $\{\omega_n^i\pi_n^j : 0 \leq i, j \leq n-1\}$, or else uniquely in the form $z = \alpha_0 + \alpha_1\pi + \cdots + \alpha_{n-1}\pi_n^{n-1}$ with $\alpha_i \in \mathbb{Z}_2(\zeta)$.

The Maximal Order

We present in particular some results for $\Delta_3 = \mathbb{Z}_2[\omega, \pi]$ with

$$\omega = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix} \qquad \pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

where ζ is a 7^{th} root of unity.
In addition to the relations $\pi\omega = \omega^2\pi$ and $\pi^3 = 2I_3$, we also work with the convention that

$$\zeta + \zeta^2 + \zeta^4 \equiv 0 \pmod{2} \quad \text{and} \quad \zeta^3 + \zeta^5 + \zeta^6 \equiv 1 \pmod{2}.$$

The valuation in Δ_3 is described by $\nu(z) = \nu_2(\det(z))$ for $z \in \Delta_3$ realized as a matrix, where ν_2 denotes the 2-adic valuation.

Conjugacy Classes mod π

Looking at all elements of $\Delta_3 = \mathbb{Z}_2[\omega, \pi]$ modulo π, we obtain four conjugacy classes:

$$\begin{aligned} T &= \{z \in \Delta_3 : z \equiv 0 \pmod{\pi}\} \\ T+1 &= \{z \in \Delta_3 : z \equiv I_3 \pmod{\pi}\} \\ S &= \{z \in \Delta_3 : z \equiv \omega \text{ or } \omega^2 \text{ or } \omega^4 \pmod{\pi}\} \\ S+1 &= \{z \in \Delta_3 : z \equiv \omega^3 \text{ or } \omega^6 \text{ or } \omega^5 \pmod{\pi}\} \\ &= \{z \in \Delta_3 : z \equiv \omega + I_3 \text{ or } \omega^2 + I_3 \text{ or } \omega^4 + I_3 \pmod{\pi}\} \end{aligned}$$

Conjugacy Classes mod π^2

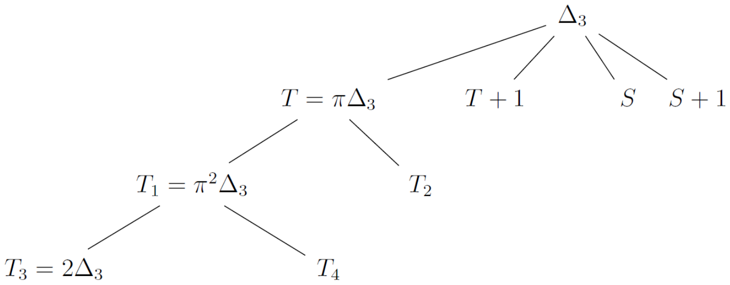
We can break the set T down further by considering conjugacy classes modulo π^2:

$$T_1 = \{z \in \Delta_3 : z \equiv 0 \pmod{\pi^2}\} = \pi^2\Delta$$
$$T_2 = \{z \in \Delta_3 : z \equiv \omega^i \pi \pmod{\pi^2} \text{ for some } 0 \leq i \leq 6\}$$

The set T_1 can be broken down further still by looking at conjugacy classes modulo $\pi^3 = 2$:

$$T_3 = \{z \in \Delta_3 : z \equiv 0 \pmod{\pi^3}\} = 2\Delta$$
$$T_4 = \{z \in \Delta_3 : z \equiv \omega^i \pi^2 \pmod{\pi^3} \text{ for some } 0 \leq i \leq 6\}$$

From this analysis, we obtain the following tree of subsets of Δ_3:



These sets all satisfy the necessary lemmas pertaining to shuffles of ν-sequences, and so we can derive a formula for α_{Δ_3} that depends only on itself, α_S, α_{T_2}, and α_{T_4}.

Characteristic Polynomials

The tree of subsets and the lemmas show us that the ν-sequence of Δ_3 is recursively defined and also depends on the ν-sequences of S, T_2, T_4.

It remains to determine the ν-sequences for these sets, and to do so, it is useful to describe them in terms of their characteristic polynomials.

Given a 3×3 matrix A, we define the characteristic polynomial of A to be

$$x^3 - Tr(A)x^2 + \beta(A)x - \det(A)$$

where $Tr(A)$ and $\det(A)$ are the usual trace and determinant of a 3×3 matrix, and $\beta(A)$ is defined in terms of the 2×2 minors of A.

Lemma

$$
\begin{aligned}
S &= \{z \in \Delta_3 : Tr(z) \equiv 0 \ (mod\ 2),\ \beta(z) \equiv 1 \ (mod\ 2), \det(z) \equiv 1 \ (mod\ 2)\} \\
T_2 &= \{z \in \Delta_3 : Tr(z) \equiv 0 \ (mod\ 2),\ \beta(z) \equiv 0 \ (mod\ 2), \det(z) \equiv 2 \ (mod\ 4)\} \\
T_4 &= \{z \in \Delta_3 : Tr(z) \equiv 0 \ (mod\ 2),\ \beta(z) \equiv 0 \ (mod\ 4), \det(z) \equiv 4 \ (mod\ 8)\}
\end{aligned}
$$

We can determine some useful facts about the valuation of certain polynomials within S, T_2, and T_4, with the goal of establishing these as the minimal polynomials within their respective sets. This process is analogous to the one presented in Evrard and Johnson [3] and Johnson [4].

A Polynomial in T_2

Recall that

$$T_2 = \{z \in \Delta_3 : Tr(z) \equiv 0 \pmod 2,\ \beta(z) \equiv 0 \pmod 2, \det(z) \equiv 2 \pmod 4\}.$$

Let us define the function

$$\psi = (\psi_1, \psi_2, \psi_3) : \mathbb{Z}_{\geq 0} \to 2\mathbb{Z}_{\geq 0} \times 2\mathbb{Z}_{\geq 0} \times (2 + 4\mathbb{Z}_{\geq 0})$$

$$\psi(n) = \left(2 \sum_{i \geq 0} n_{3i+1} 2^i, 2 \sum_{i \geq 0} n_{3i} 2^i, 2 + 4 \sum_{i \geq 0} n_{3i+2} 2^i \right)$$

where $n = \sum_{i \geq 0} n_i 2^i$ is the expansion of n in base 2. Let

$$g_n(x) = \prod_{k=0}^{n-1} \left(x^3 - \psi_1(k) x^2 + \psi_2(k) x - \psi_3(k) \right).$$

Lemma

If $z \in T_2$ then

$$\nu(g_n(z)) \geq 4n + \sum_{i > 0} \left\lfloor \frac{n}{2^i} \right\rfloor.$$

The polynomials constructed in the previous slide will be the minimal polynomial of a sequence of elements in T_2, which then suggests that this sequence extends to a ν-ordering. The associated ν-sequence will be the valuation of these polynomials, which we have calculated.

This method of creating minimal polynomials based on the characteristic polynomial that defines a conjugacy class within Δ_3 can be extended to any subset S of a maximal order Δ_n sitting in $M_n(\mathbb{Q}_2)$ that is closed under conjugation. However, the practical use of the construction comes from the fact that it is possible to achieve a known minimum when taking the valuation of the polynomials generated.

Extension to General n

For any valuation ν, if the valuation of n terms $a_1, \ldots, a_n$ produces a complete set of residues modulo n, then it must be the case that $\nu(a_1 + \cdots + a_n) = \min_{1 \leq i \leq n} \nu(a_i)$. This fact is applied in the valuation of the polynomial

$$f(z) = z^n - \phi_1(k)z^{n-1} + \phi_2(k)z^{n-2} + \cdots + (-1)^n \phi_n(k)$$

with $z \in S \subseteq \Delta_n$ to show that a minimum for $\nu(f)$ can be determined with certainty only when $\gcd(n, \nu(z)) = 1$.

In particular, if $n = q$ is a prime, then a polynomial construction such as that of T_2 in the 3×3 case (given in detail for the 2×2 case in [3] and [4]) will be possible for all conjugacy classes in the maximal order Δ_q.

The construction will also work for some subsets of Δ_n when n is composite, in particular for conjugacy classes modulo π^j where $\gcd(j, n) = 1$. It remains to see what adjustments must be made to this construction in the case where n is composite, and if there is any difference between the case where n is a power of a prime or n is squarefree.

Structure of Δ_4

We now consider $\Delta_4 = \mathbb{Z}_2[\omega, \pi]$ with

$$\omega = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 \\ 0 & 0 & 0 & \zeta^8 \end{pmatrix} \qquad \pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

where ζ is a 15^{th} root of unity.

In addition to the relations $\pi\omega = \omega^2\pi$ and $\pi^4 = 2I_4$, we also work with the convention that

$$\zeta^3 + \zeta^4 + \zeta^7 \equiv 0 \pmod 2 \quad \text{and} \quad \zeta + \zeta^5 + \zeta^8 \equiv 1 \pmod 2 .$$

As previously, the valuation in Δ_4 is described by $\nu(z) = \nu_2(\det(z))$ for $z \in \Delta_4$ realized as a matrix, where ν_2 denotes the 2-adic valuation.

Conjugacy Classes modulo *pi*

Looking at all elements of $\Delta_4 = \mathbb{Z}_2[\omega, \pi]$ modulo π, we obtain six conjugacy classes:

$$\begin{aligned}
T &= \{z \in \Delta_4 : z \equiv 0 \pmod{\pi}\} = \pi\Delta \\
T+1 &= \{z \in \Delta_4 : z \equiv I_4 \pmod{\pi}\} \\
S_1 &= \{z \in \Delta_4 : z \equiv \omega \text{ or } \omega^2 \text{ or } \omega^4 \text{ or } \omega^8 \pmod{\pi}\} \\
S_2 &= \{z \in \Delta_4 : z \equiv \omega^7 \text{ or } \omega^{11} \text{ or } \omega^{13} \text{ or } \omega^{14} \pmod{\pi}\} \\
S_3 &= \{z \in \Delta_4 : z \equiv \omega^3 \text{ or } \omega^6 \text{ or } \omega^9 \text{ or } \omega^{12} \pmod{\pi}\} \\
S_4 &= \{z \in \Delta_4 : z \equiv \omega^5 \text{ or } \omega^{10} \pmod{\pi}\}
\end{aligned}$$

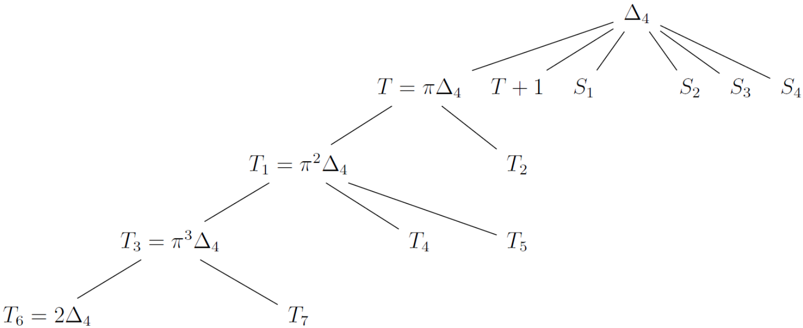
We can break down the set T further into subsets:

$$T_1 = \{z \in \Delta_4 : z \equiv 0 \pmod{\pi^2}\} = \pi^2 \Delta_4$$
$$T_2 = \{z \in \Delta_4 : z \equiv \omega^i \pi \pmod{\pi^2} \text{ for some } 0 \leq i \leq 14\}$$

$$T_3 = \{z \in \Delta_4 : z \equiv 0 \pmod{\pi^3}\} = \pi^3 \Delta_4$$
$$T_4 = \{z \in \Delta_4 : z \equiv \omega^i \pi^2 \pmod{\pi^3} \text{ for some } i \equiv 0 \pmod{3}\}$$
$$T_5 = \{z \in \Delta_4 : z \equiv \omega^i \pi^2 \pmod{\pi^3} \text{ for some } i \not\equiv 0 \pmod{3}\}$$

$$T_6 = \{z \in \Delta_4 : z \equiv 0 \pmod{\pi^4}\} = \{z \in \Delta_4 : z \equiv 0 \pmod{2}\} = 2\Delta_4$$
$$T_7 = \{z \in \Delta_4 : z \equiv \omega^i \pi^3 \pmod{\pi^4} \text{ for some } 0 \leq i \leq 14\}$$

From this analysis, we obtain the following tree of subsets of Δ_4:



ν-sequence of Δ_4

The ν-sequence of Δ_4 will be recursively defined, and will also depend on the ν-sequences of the S_i, T_2, T_4, T_5, and T_7.

For each $z \in S_i$ we have $\nu(z) = 0$, for $z \in T_2$ we have $\nu(z) = 1$, for $z \in T_4$ and $z \in T_5$ we have $\nu(z) = 2$, and for $z \in T_7$ we have $\nu(z) = 3$. Since our aforementioned construction involving taking the valuation of products of characteristic polynomials works when $\gcd(n, \nu(z)) = 1$, we will be able to use this method for computing the ν-sequences of the S_i for $i = 1, 2, 3$, T_2, and T_7.

We will encounter problems for S_4 since the characteristic polynomial of its elements modulo 2 is reducible, and for T_4 and T_5 because the valuation of elements in the set are not relatively prime to the dimension.

A Potential Polynomial in T_5

Let us define the function

$$\phi = (\phi_1, \phi_2, \phi_3, \phi_4) : \mathbb{Z}_{\geq 0} \to 2\mathbb{Z}_{\geq 0} \times (2 + 4\mathbb{Z}_{\geq 0}) \times 4\mathbb{Z}_{\geq 0} \times (4 + 8\mathbb{Z}_{\geq 0})$$

$$\phi(n) = \left(2\sum_{i\geq 0} n_{4i}2^i, 2 + 4\sum_{i\geq 0} n_{4i+2}2^i, 4\sum_{i\geq 0} n_{4i+1}2^i, 4 + 8\sum_{i\geq 0} n_{4i+3}2^i \right)$$

where $n = \sum_{i\geq 0} n_i 2^i$ is the expansion of n in base 2. Let $z \in T_5$, let $k \geq 0$, and let

$$f_z(k) = z^4 - \phi_1(k)z^3 + \phi_2(k)z^2 - \phi_3(k)z + \phi_4(k) \ .$$

Then

$$\nu(f_z(k)) \geq \begin{cases} 10 + \nu_2(m-k) & \text{if } \nu_2(m-k) \equiv 0 \pmod 2 \\ 9 + \nu_2(m-k) & \text{if } \nu_2(m-k) \equiv 1 \pmod 2 \end{cases}$$

where $m \in \mathbb{Z}$ is chosen so that $f(m)$ is the characteristic polynomial of $z \in T_5$.

$$\nu(f_z(k)) \geq \begin{cases} 10 + \nu_2(m-k) & \text{if } \nu_2(m-k) \equiv 0 \pmod 2 \\ 9 + \nu_2(m-k) & \text{if } \nu_2(m-k) \equiv 1 \pmod 2 \end{cases}$$

Note that due to the nature of the set T_5, we will not have any cancellation of terms when evaluating $\nu(f_z(k))$. This means that equality can be achieved in the expression above, and so too is the case for products of such polynomials $f_z(k)$, as we saw in the 3×3 case. Therefore, we are still able to use this construction to establish a ν-sequence for T_5.

The Case of T_4

For $z \in T_4$, the result in constructing potential minimal polynomials is the same as for T_5:

$$\nu(f_z(k)) \geq \begin{cases} 10 + \nu_2(m-k) & \text{if } \nu_2(m-k) \equiv 0 \pmod 2 \\ 9 + \nu_2(m-k) & \text{if } \nu_2(m-k) \equiv 1 \pmod 2 \end{cases}$$

However, in the case of T_4, it is possible to choose elements in the set such that elements cancel when computing the valuation of a polynomial $f_z(k)$.

This means that we *cannot* guarantee equality in the above expression, and our inequality becomes strict when we consider products of such polynomials $f_z(k)$. A different method of approach is necessary for T_4.

Next Steps

We can view Δ_2 as being embedded in Δ_4. In Δ_2, the subset denoted T_1 is defined by

$$T_1 = \{z \in \Delta_2 : Tr(z) \equiv 0 \pmod 2, N(z) \equiv 2 \pmod 4\}$$

where characteristic polynomials are denoted as $x^2 - Tr(z)x + N(z)$. The characteristic polynomial of an element of $T_1 \subseteq \Delta_2$, when squared, has the same form as expected for the characteristic polynomial of an element in $T_4 \subseteq \Delta_4$.

We may be able to learn more about the ν-sequence of T_4 by looking at the squares of polynomials in Δ_2 and noting the relationship with the denominator.

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