

A Linear Algebra Problem Related to Legendre Polynomials

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• Introduction: Legendre Polynomials

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These polynomials come up throughout numerous areas of physics, most notably in multipole expansions in electrodynamics and in certain solutions of the Schrödinger equation in quantum mechanics.

The first few Legendre polynomials are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2} (3x^2 - 1)$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3)$$

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From the generating function it can be shown that they also satisfy the three term recurrence relation

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$$

An explicit formula for the Legendre polynomials is given by

$$P_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k$$

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$$\int_{-1}^1 P_n(x)P_m(x)dx = \begin{cases} \frac{2}{2n+1} & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

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An alternate derivation of the Legendre polynomials involves using the Gram-Schmidt method on the polynomials $\{1, x, x^2, \dots\}$ under the above inner product.



Figure: Watercolor of Adrien-Marie Legendre. This is the only known portrait of Legendre and for nearly 200 years a different portrait was mistaken to be Legendre, but was actually of a politician, Louis Legendre.

• Statement of the Original Problem

Find $f(x)$ such that

$$g\left(\frac{1}{2}\right) = \int_0^1 f(x)g(x)dx$$

where $f(x)$ and $g(x)$ are polynomials of degree ≤ 2 .

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Increase to degree 4 and we have

$$f(x) = \frac{945}{4}x^4 - \frac{945}{2}x^3 + \frac{1155}{4}x^2 - \frac{105}{2}x + \frac{15}{8} \text{ as the solution.}$$

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Degree 5 also has $f(x) = \frac{945}{4}x^4 - \frac{945}{2}x^3 + \frac{1155}{4}x^2 - \frac{105}{2}x + \frac{15}{8}$ as the solution.

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This pattern continues for higher degrees. Why is this happening? To answer this we change the original problem to a more general one.

• Generalizing the Problem

Find $f_n(x)$ such that

$$g(c) = \int_0^1 f_n(x)g(x)dx$$

where $g(x)$ and $f_n(x)$ are polynomials of degree $\leq n$.

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where $g(x)$ and $f_n(x)$ are polynomials of degree $\leq n$.

Now let us write our question in terms of the following proposition.

Proposition

Let $f_n(x)$ be as previously defined. Then when $c = \frac{1}{2}$ we have $f_{2m+1}(x) = f_{2m}(x)$ for $m \in \mathbb{N}$.

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Now solving the more general problem leads us to the following matrix multiplication.

$$\begin{bmatrix}
 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\
 \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\
 \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1}
 \end{bmatrix}
 \begin{bmatrix}
 a_0 \\
 a_1 \\
 a_2 \\
 \vdots \\
 a_n
 \end{bmatrix}
 =
 \begin{bmatrix}
 1 \\
 c \\
 c^2 \\
 \vdots \\
 c^n
 \end{bmatrix}$$

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Now write this as

$$H\underline{a} = \underline{c}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 \\ c \\ c^2 \\ \vdots \\ c^n \end{bmatrix}$$

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The choice of H is because matrices of this form ($H_{i,j} = \frac{1}{i+j-1}$) are known as Hilbert matrices.

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$$(H_{i,j})^{-1} = (-1)^{i+j-1} (i+j-1) \binom{n+i}{i+j-1} \binom{n+j}{i+j-1} \binom{i+j-2}{i-1}^2.$$

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Before continuing we need a definition.

Definition

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These polynomials determine the coefficients of $f_n(x)$. That is,

$$h_{n,i}(c) = a_{i-1}, \text{ or } f_n(x) = \sum_{k=1}^{n+1} h_{n,k}(c)x^{k-1}$$

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Let's do an example to clarify.

Example: If $n = 2$, then

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

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Using our formula,

$$H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

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So we have $n = 2$ and can see that $1 \leq i \leq 3$. Thus

$$h_{2,1}(c) = 9 - 36c + 30c^2,$$

$$h_{2,2}(c) = -36 + 192c - 180c^2,$$

$$h_{2,3}(c) = 30 - 180c + 180c^2,$$

$$f_n(x) = h_{2,1}(c) + h_{2,2}(c)x + h_{2,3}(c)x^2$$

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If n is odd, and $f_n(x) = f_{n-1}(x)$ when $c = \frac{1}{2}$, then the degree of $f_n(x)$ is $n - 1$. This means that $a_n = 0$ in $f_n(x)$.

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This would correspond to the polynomial formed by the bottom row of H^{-1} having a root at $c = \frac{1}{2}$. Using our definition, this can be written as $h_{n,n+1}(c)$ having a root at $c = \frac{1}{2}$.

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Using the formula for H^{-1} we can write $h_{n,n+1}(c)$ as

$$h_{n,n+1}(c) = \sum_{j=1}^{n+1} (-1)^{n+j+1} (n+j) \binom{2n+1}{n+j} \binom{n+j}{n+j} \binom{n+j-1}{n}^2 c^{j-1}$$

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All we need now is a definition to solve our problem.

Definition

The shifted Legendre polynomials, denoted $\tilde{P}_n(x)$, are given by

$$\tilde{P}_n(x) = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} x^k.$$

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Looking back at our expression for $h_{n,n+1}(c)$,

$$h_{n,n+1}(c) = (-1)^n (2n+1) \binom{2n+1}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} c^k,$$

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we can now see that $h_{n,n+1}(c)$ is just a multiple of $\tilde{P}_n(c)$.

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The Legendre polynomials are known to have $x = 0$ as a root when their degree is odd. Therefore, the shifted Legendre polynomials must have a root at $x = \frac{1}{2}$ when their degree is odd.

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The shifted Legendre polynomials have a root at $\frac{1}{2}$ when their degree is odd.

Therefore, if n is odd and $c = \frac{1}{2}$, then $a_n = 0$. This ends up forcing $f_n(x) = f_{n-1}(x)$, thus answering our question.

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If $h_{n,n+1}(c)$ is always just a multiple of a shifted Legendre polynomial, what do other rows correspond to?

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That is, how does $h_{n,1}(c)$ change as we change n ? $h_{n,2}(c)$? etc.

- **Another Approach to the Problem and the Other Rows of H^{-1}**

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Theorem

(Riesz Representation Theorem) *If we have some finite dimensional vector space, V , and some linear functional ϕ on V , then there is a unique vector $u \in V$ such that*

$$\phi(v) = \langle v, u \rangle$$

for all $v \in V$.

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We can interpret our problem in terms of this theorem. The integral is an inner product, $g(x)$ corresponds to v , $f_n(x)$ corresponds to u , and evaluation at c is a linear functional.

This theorem has the consequence of allowing us to write

$$\begin{aligned} f_n(x) &= \sum_{k=0}^n \frac{\tilde{P}_k(c)\tilde{P}_k(x)}{\int_0^1 \tilde{P}_k(x)^2 dx} \\ &= \sum_{k=0}^n (2k+1)\tilde{P}_k(c)\tilde{P}_k(x) \end{aligned}$$

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It should be noted that this expression for $f_n(x)$ shows us that it is actually a familiar concept in the study of orthogonal polynomials.

In this form $f_n(x)$ would be called the kernel of the shifted Legendre polynomials. Therefore what I am studying can be interpreted as looking at how the coefficients of this kernel change with n , and with c .

Moving back to $h_{n,i}(c)$, we can use the expression from the previous slide:

$$f_n(x) = \sum_{k=0}^n (2k + 1) \tilde{P}_k(c) \tilde{P}_k(x)$$

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$$h_{n,i}(c) = \sum_{k=i-1}^n (-1)^{k+i-1} \binom{k}{i-1} \binom{k+i-1}{i-1} (2k+1) \tilde{P}_k(c)$$

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Using this equation, I wanted to find a generating function for these polynomials.

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Since the Legendre Polynomials satisfy a recurrence relation, the shifted Legendre polynomials must also.

$$(n + 1)\tilde{P}_{n+1}(x) = (2n + 1)(2x - 1)\tilde{P}_n(x) - n\tilde{P}_{n-1}(x)$$

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Combining the two expressions, it can be shown that

$$h_{n,1}(c) = \frac{(-1)^n(n+1)}{2c} \left(\tilde{P}_n(c) + \tilde{P}_{n+1}(c) \right)$$

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If $i = 1$ we have

$$h_{n,1}(c) = \sum_{k=0}^n (-1)^k (2k+1) \tilde{P}_k(c).$$

Since the Legendre Polynomials satisfy a recurrence relation, the shifted Legendre polynomials must also.

$$(n+1)\tilde{P}_{n+1}(x) = (2n+1)(2x-1)\tilde{P}_n(x) - n\tilde{P}_{n-1}(x)$$

Combining the two expressions, it can be shown that

$$h_{n,1}(c) = \frac{(-1)^n(n+1)}{2c} \left(\tilde{P}_n(c) + \tilde{P}_{n+1}(c) \right)$$

Using this we can find a generating function for $h_{n,1}(c)$.

Rearranging, multiplying by x^{n+1} , and summing over n yields

$$\sum_{n=0}^{\infty} \frac{h_{n,1}(c)x^{n+1}}{n+1} = - \sum_{n=0}^{\infty} \frac{1}{2c} \left(\tilde{P}_n(c)(-x)^{n+1} + \tilde{P}_{n+1}(c)(-x)^{n+1} \right)$$

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Now we take the derivative with respect to x of both sides which gives us the generating function.

Let $\mathcal{H}_1(x)$ be the generating function for $h_{n,1}(c)$.

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Then we have from the previous slide

$$\begin{aligned}\mathcal{H}_1(x) &= -\frac{1}{2c} \frac{d}{dx} \left(\frac{1-x}{\sqrt{1+2(2c-1)x+x^2}} - 1 \right) \\ &= \frac{1+x}{(1+2(2c-1)x+x^2)^{\frac{3}{2}}}\end{aligned}$$

Now using our expression for $\mathcal{H}_1(x)$, along with

$$h_{n,i}(c) = \sum_{k=i-1}^n (-1)^{k+i-1} \binom{k}{i-1} \binom{k+i-1}{i-1} (2k+1) \tilde{P}_k(c)$$

we can find an expression for the generating function of any $h_{n,i}(c)$, denoted $\mathcal{H}_i(x)$.

Sparing the details of the calculation, as it is more complicated but similar to the derivation of $\mathcal{H}_1(x)$, the final final result is given by

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So we have accomplished our goal of finding the generating function for the polynomials $h_{n,i}(c)$.

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The part I want to show however, is the polynomial in the numerator.

For $\mathcal{H}_1(x)$ the polynomial would just be 1.

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And for $\mathcal{H}_3(x)$

$$\begin{aligned} & (c^2 - c + \frac{1}{6})x^4 + (-\frac{4}{3}c^3 + 2c^2 + \frac{2}{3}c - \frac{2}{3})x^3 \\ & \quad - \frac{1}{3}(c^2 - c + 3)(c^2 - c - 1)x^2 \\ & + (-\frac{4}{3}c^3 + 2c^2 + \frac{2}{3}c - \frac{2}{3})x + (c^2 - c + \frac{1}{6}). \end{aligned}$$

With the generating functions found, we can now draw a comparison to a more general set of orthogonal polynomials, the Gegenbauer polynomials (also known as the ultraspherical polynomials), denoted $C_n(t)$.

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These polynomials generalize the Legendre polynomials, as well as other types of orthogonal polynomials. In turn, the Gegenbauer polynomials are generalized by what are known as the Jacobi polynomials.

These polynomials can be defined through the generating function

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Comparing this with the generating functions for the $h_{n,i}(c)$, we see that they are actually quite similar. For instance,

$$\mathcal{H}_2(x) = \frac{12x(1+x) \left((c - \frac{1}{2})x^2 - (c^2 - c - 1)x + c - \frac{1}{2} \right)}{(1 + 2(2c - 1)x + x^2)^{\frac{7}{2}}}$$

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While they initially may not look that much alike, the denominator is what really matters here.

The Gegenbauer polynomials follow the recurrence relation

$$C_n^\alpha(t) = \frac{1}{n} (2t(n + \alpha - 1)C_{n-1}^\alpha(t) - (n + 2\alpha - 2)C_{n-2}^\alpha(t))$$

where $C_0^\alpha(t) = 1$ and $C_1^\alpha(t) = 2\alpha t$.

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From this we can find a recurrence relation for the $h_{n,i}(c)$, but we need to further study the polynomials arising in the numerators of the generating functions.

• Other Results

Now I will quickly state some other results and properties of these polynomials.

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$$\int_0^1 h_{n,i}(c) c^n dc = \begin{cases} 1 & \text{if } i = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

In other words, integrating $h_{n,i}(c)$ against another polynomial in c of degree at least n , will be equal to the coefficient of c^{i-1} of the polynomial.

Another representation for these polynomials is given by

$$h_{n,i}(c) = (-1)^i i \binom{n+i}{i} \binom{n+1}{i} {}_3F_2(-n, n+2, i; 1, i+1; c)$$

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$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

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Where $(a)_n = a(a+1)\dots(a+n-1)$ is what is known as a Pochhammer symbol, or rising factorial. So,

$$h_{n,i}(c) = (-1)^i i \binom{n+i}{i} \binom{n+1}{i} \sum_{k=0}^{\infty} \frac{(-n)_k (n+2)_k (i)_k}{(1)_k (i+1)_k} \frac{z^k}{k!}$$

This just gives us another avenue through which these polynomials which can be investigated. For instance given the hypergeometric representation I was able to prove the following identity.

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$$\int_0^1 {}_3F_2(-n, n+2, i; 1, i+1; c) {}_3F_2(-m, m+2, i; 1, i+1; c) dc$$

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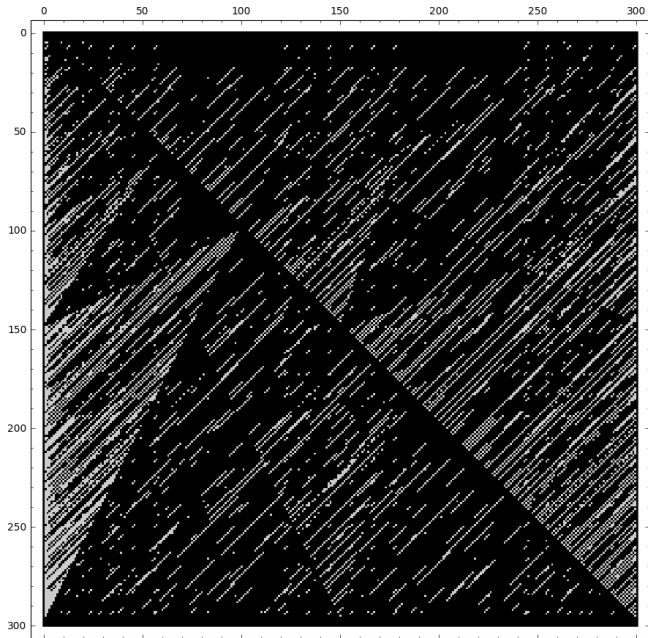
$$\int_0^1 {}_3F_2(-n, n+2, i; 1, i+1; c) {}_3F_2(-m, m+2, i; 1, i+1; c) dc$$

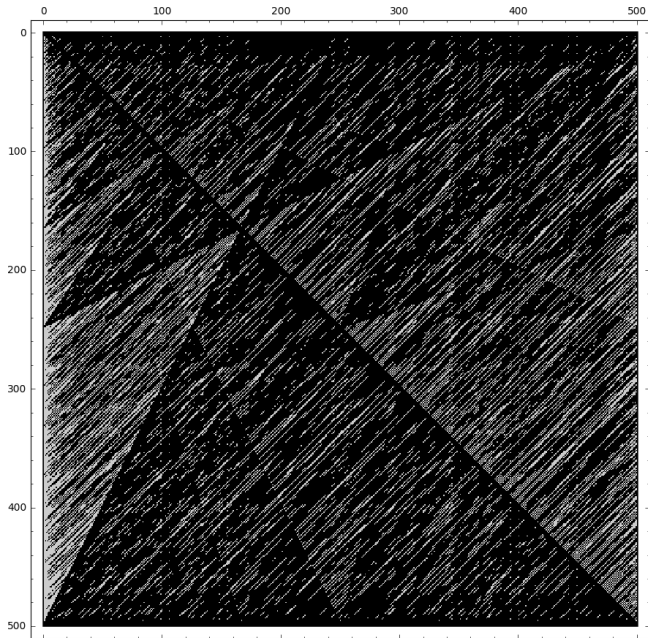
$$= \frac{i^2 \Gamma(n+i+1) \Gamma(m+2-i)}{(2i-1)(n+1)(m+1) \Gamma(n+2-i) \Gamma(m+i+1)}$$

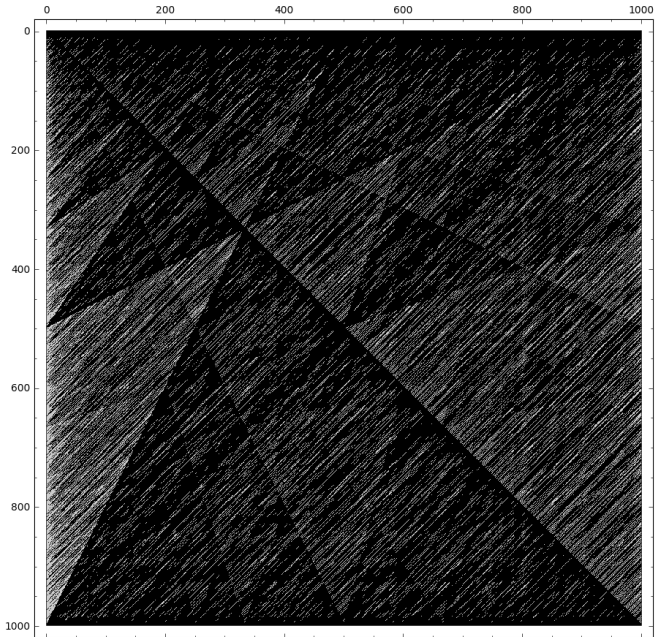
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This leads to the following interesting images.







Thanks for listening.