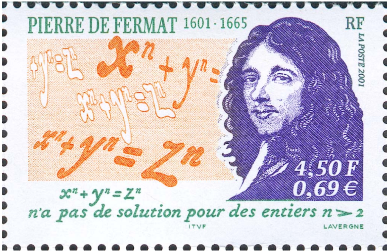
Fermat's Last Theorem: A very brief outline of its proof

Karl Dilcher

Supplementary class to MATH 4070/5070

November 30, 2018



“I have discovered a truly marvelous proof that it is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. This margin is too narrow to contain it."

– Fermat

teruallo quadratorum, & Canones iidem hîc etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 - 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. - 4. ipse igitur quadratus erit 4 Q. + 16. - 16 N. hæc æquabuntur vnitatibus 16 - 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$. & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

ΤΟΝ ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους. ἐπιτετάχθω δὴ τὸν ις διελεῖν εἰς δύο τετραγώνους. καὶ τετάχθω ὁ πρῶτος δυνάμεως μιᾶς. δεήσει ἄρα μονάδας ις λείψει δυνάμεως μιᾶς ἴσας εἶναι τετραγώνῳ. πλάσσω τὸν τετράγωνον ἀπὸ ςς. ὅσων δή ποτε λείψει τοσούτων μο ὅσων ἐστὶν ἡ τῶν ις μο πλευρά. ἔστω ςς β λείψει μο δ. αὐτὸς ἄρα ὁ τετράγωνος ἔσται δυνάμεων δ μο ις [λείψει ςς ις] ταῦτα ἴσα μονάσι ις λείψει δυνάμεως μιᾶς. κοινὴ προσκείσθω ἡ λεῖψις, καὶ ἀπὸ ὁμοίων ὅμοια. δυνάμεις ἄρα ε ἴσαι ἀριθμοῖς ις. καὶ γίνεται ὁ ἀριθμὸς ις πέμπτων. ἔσται ὁ μὲν σνς εἰκοστοπέμπτων. ὁ δὲ ρμδ εἰκοστοπέμπτων, καὶ οἱ δύο συντεθέντες ποιοῦσι υ εἰκοστόπεμπτα, ἤτοι μονάδας ις. καὶ ἔστιν ἑκάτερος τετράγωνος.

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Given an integer $n \geq 3$, there are no $a, b, c \in \mathbb{Z}$ such that

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(2) Early attempts, up to 1847:
$n = 3, 4, 5, 7, 14$; some criteria.

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Ernst Eduard Kummer
1810 - 1893

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(7) Further developments, refinements, extensions.

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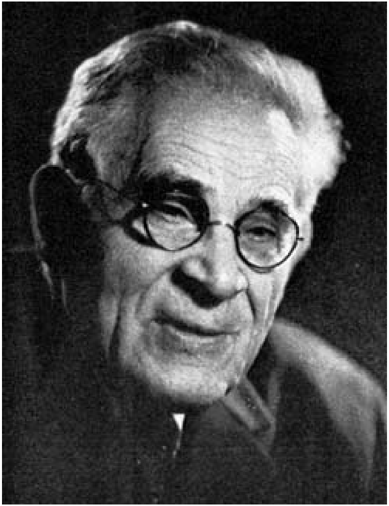
So, let's have a brief look at Mordell's Conjecture.



Louis Joel Mordell
1888 - 1972

In 1953 Mordell retired from the Sadleirian Chair but he most certainly did not retire from mathematics; **almost half** of Mordell's 270 publications appeared **after his retirement**. Nor did retirement mean that he lived a quiet life at his home in Cambridge. On the contrary he delighted in accepting appointments as Visiting Professor (in places such as Toronto, Ghana, Nigeria, **Mount Allison**, Colorado, Notre Dame and Arizona), delighted in adding yet another university to the list of places at which he had been invited to speak (with a final total of around 190), and delighted in sharing his enjoyment of mathematics with as many young people as he could.

– http://www-history.mcs.st-and.ac.uk/

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Connection with Fermat:
Rewrite $a^n + b^n = c^n$ as

$$x^n + y^n - 1 = 0.$$

Does it have *rational* solutions?

L. J. Mordell's idea:
Given a polynomial equation $Q(x, y) = 0$,
look at all its *complex* solutions.

This is related to a surface (a “compact Riemann surface"), so the genus makes sense, and will be called the “genus of the algebraic curve" defined by $Q(x, y) = 0$.

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A polynomial equation $Q(x, y) = 0$ with rational coefficients and genus $g \geq 2$ has only finitely many solutions.

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Hence:
Fermat’s equation has at most finitely many solutions for $n \geq 4$.

Gerd Faltings
1954 –

Faltings received the 1986 Fields Medal for this achievement.

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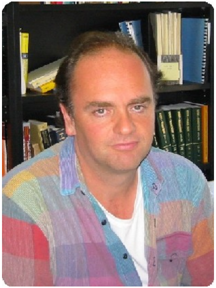

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Some fundamental new ideas were introduced by
Y. Hellegouarch (1975) and G. Frey (1982).

Idea: Suppose that FLT is false, i.e., suppose there exist nonzero $a, b, c \in \mathbb{Z}$, pairwise coprime, such that

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Since

$$b^p + a^p = c^p \quad \text{and} \quad a^p + (-c)^p = (-b)^p$$

are also solutions, we may as well assume that we have $a \equiv -1 \pmod 4$ and b is even.

Elliptic curves have a number of “invariants"; one of them is the *discriminant*, defined by

$$(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2,$$

where x_1, x_2, x_3 are the roots of the RHS in (1).

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With $x_1 = 0$, $x_2 = -a^p$, $x_3 = b^p$, the discriminant is

$$(0 - (-a^p))^2(-a^p - b^p)^2(b^p - 0)^2 = a^{2p}b^{2p}c^{2p},$$

(recall that $a^p + b^p = c^p$.)

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Wiles picked up on this and set out to prove the TSW conjecture, working in isolation for the following 7 years.

Gerhard Frey
1944–

Ken Ribet
1947–

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Lemma: Every Frey curve is semistable.

Proof: If $\ell \mid a^{2p}b^{2p}c^{2p}$, then ℓ divides only one of a^p, b^p, c^p since they are coprime. The result follows from the roots being $0, a^p, -b^p$ (recall $c^p = a^p + b^p$).

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Question: What does “modular" mean??

5. Finite Fields, Projective Plane

Main idea:
Consider elliptic curves not over $\mathbb{Q}$,
but over a finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, i.e., mod p,
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Example: $y^2 = x^3 - 5$. Discriminant is

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The primes of bad reduction are multiplied together (with certain exponents) to give the *conductor* N of the curve.

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Roughly speaking: The collection of all points

$$(x, y, z) \in \mathbb{R}^3 \setminus (0, 0, 0),$$
$$(x, y, z) \sim (ax, ay, az).$$

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The “point at infinity" that serves as identity in the elliptic curve group is then $\mathcal{O} = (0, 1, 0)$.

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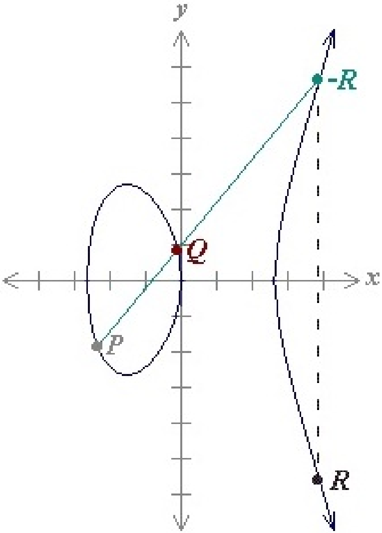

$$y^2 = x^3 - 7x$$

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When $z = 0$: Given $y = 1$, exactly one x
When $z = 1$: For each y, exactly one x
Total: $p + 1$ solutions.

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Example: $y^2 = x^3 + 22$ over $\mathbb{F}_5$.
Homogenize: $y^2 z = x^3 + 22z^3$.
Reduce modulo 5: $y^2 z = x^3 + 2z^3$.
Find solutions (“trial and error"):

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Reduce modulo 5: $y^2 z = x^3 + 2z^3$.
Find solutions ("trial and error"):

$$(x, y, z) = (0, 1, 0), \qquad \begin{array}{l} (2, 0, 1) \\ (3, 2, 1) \\ (3, 3, 1) \\ (4, 1, 1) \\ (4, 4, 1) \end{array}$$

Hence $b_5 = 6$, and thus $a_5 = 0$.

The surprise now is:
These numbers a_p will appear in a different, seemingly unrelated setting as Fourier coefficients of certain functions.

6. Modular Forms

Recall from complex analysis:

$$g(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0,$$

$(a, b, c, d \in \mathbb{C})$ is called a *Möbius map*.

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The **modular group** is the group of all Möbius maps

$$g(z) = \frac{az+b}{cz+d},$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

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(this includes straight lines in $\mathbb{C}$)
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The **modular group** is the group of all Möbius maps

$$g(z) = \frac{az+b}{cz+d},$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

It maps the upper-half plane
$\mathbb{H} = \{x + iy \mid y > 0\}$ into itself.

The modular group can (basically) be identified with the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

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Subgroups of these turn out to be more interesting: Define

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

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We want to consider functions on $\mathbb{H}$ which “transform well" under one of the subgroups $\Gamma_0(N)$. In particular, we require that there be an integer k such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for all} \quad \gamma \in \Gamma_0(N).$$

For $\gamma = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, we have $f(z+1) = f(z)$,
so f must have a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n, \qquad q = e^{2\pi i z}.$$

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- N is called the *level* of f.

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There is a family of operators, called “Hecke operators", acting on the space of modular forms of given weight and level. Eigenvectors of Hecke operators are called *eigenforms*.

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There are other technical conditions that make a modular form a “cusp form" and a “new form".

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- of weight 2 and level N,
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Then there exists an elliptic curve with
- integer coefficients,
- conductor N,
- the a_p (as defined earlier) agreeing with the Fourier coefficients of f.

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Every elliptic curve arises in this manner;
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This proves Fermat's Last Theorem.

Andrew John Wiles (1953 –)

Wednesday 23 June 1993, around 10.30 a.m.
The Newton Institute, Cambridge, England

“Having written the theorem on the blackboard he said, ‘I will stop here’, and sat down".

It didn't quite end there . . .

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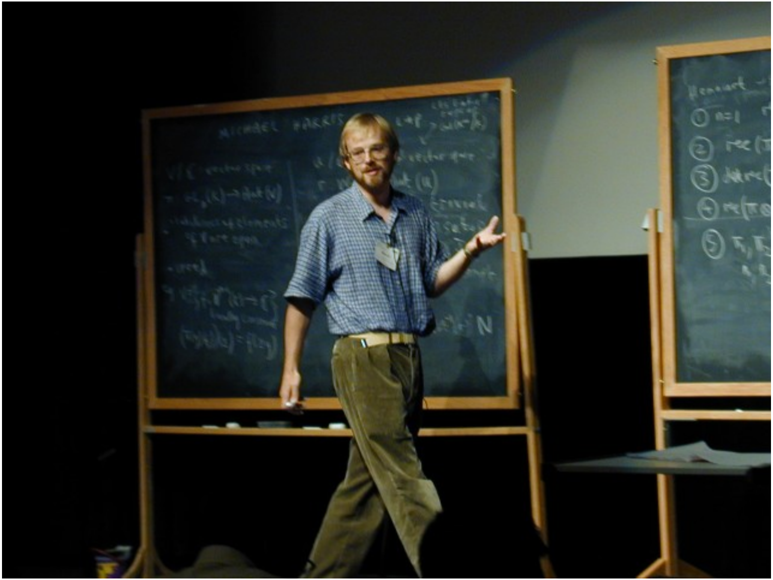
Wiles received the famous “Wolfskehl Prize" in 1997.



Richard Taylor (1962 –), Princeton, 1999

Later developments:

Recall that Wiles proved the TSW-conjecture “only" for semistable curves.

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The full conjecture was later proved by
C. Breuil, B. Conrad, F. Diamond, and R. Taylor;
announced 1999, published 2001.
(Using methods first developed by Wiles).

The generalized Fermat conjecture:
Consider

$$x^p + y^q = z^r, \qquad (p, q, r \in \mathbb{N}, 1/p + 1/q + 1/r < 1).$$

Are there solutions in integers x, y, z that have no common divisor?

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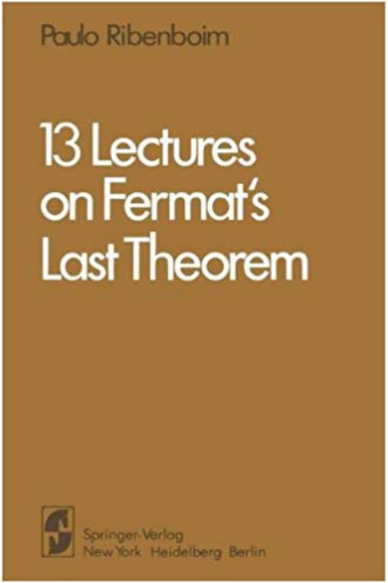
Some partial results are known.

Henri Darmon
McGill Univ.

Andrew Granville
Univ. de Montréal


Paulo Ribenboim
13 Lectures on Fermat's Last Theorem
Springer-Verlag
New York Heidelberg Berlin

Thank you

7 Kč ČESKÁ REPUBLIKA
PIERRE DE FERMAT 1670
$x^n + y^n = z^n$
ANDREW WILES 1995
2000 SVĚTOVÝ ROK MATEMATIKY

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