Orthogonal Polynomials for Bernoulli and Euler Polynomials

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Acknowledgment

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Objects

- Bernoulli numbers B_n:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!};$$

- Euler numbers E_n:

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!};$$

- Bernoulli polynomial $B_n(x)$:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!};$$

- Euler polynomial $E_n(x)$:

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!};$$

- Bernoulli polynomial of order p $B_n^{(p)}(x)$:

$$\left(\frac{t}{e^t - 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!};$$

- Euler polynomial of order p $E_n^{(p)}(x)$:

$$\left(\frac{2}{e^z + 1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{z^n}{n!};$$

$$B_n^{(1)}(x) = B_n(x);\ B_n(0) = B_n;\ E_n^{(1)}(x) = E_n(x);\ E_n(1/2) = E_n/2^n.$$

Orthogonal Polynomials

Let X be a random variable with density function $p(t)$ on $\mathbb{R}$ and with moments m_n, i.e.,

$$m_n = \mathbb{E}[X^n] = \int_{\mathbb{R}} t^n p(t) \mathrm{d}t.$$

Let $P_n(y)$ be the monic orthogonal polynomials with respect to X (or w. r. t. m_n), i.e., $\deg P_n = n$, $\mathrm{LC}[P_n] = 1$, and

$$\int_{\mathbb{R}} P_m(t) P_n(t) p(t) \mathrm{d}t = c_n \delta_{m,n} = \begin{cases} c_n, & \text{if } m = n; \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, for all $0 \leq r < n$

$$y^r P_n(y) \Big|_{y^k = m_k} = 0.$$

P_n satisfies a three-term recurrence: $P_0 = 1$, $P_1 = y - s_0$, and

$$P_{n+1}(y) = (y - s_n) P_n(y) - t_n P_{n-1}(y).$$

Example. 1. Carlitz [3, eq. 4.7] and also with Al-Salam [1, p. 93] gave the monic orthogonal polynomials, denoted by $Q_n(y)$, with respect to E_n:

$$Q_{n+1}(y) = y Q_n(y) + n^2 Q_{n-1}(y).$$

2. Touchard [16, eq. 44] computed the monic orthogonal polynomials with respect to the B_n, denoted by $R_n(y)$:

$$R_{n+1}(y) = \left(y + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y).$$

1st Task

- Find the orthogonal polynomials with respect to $B_n(x)$, denoted by $\varrho_n(y)$;
- and the orthogonal polynomials with respect to $E_n(x)$, denoted by $\Omega_n(y)$.

Namely, for $0 \leq r < n$,

$$y^r \varrho_n(y)\Big|_{y^k = B_k(x)} = 0 = y^r \Omega_n(y)\Big|_{y^k = E_k(x)}.$$

[Question] Why?

- Generalization
- Probabilistic interpretations: Letting

$$p_B(t) := \frac{\pi}{2}\operatorname{sech}^2(\pi t) \quad \text{and} \quad p_E(t) := \operatorname{sech}(\pi t), \quad (t \in \mathbb{R})$$

  we define two random variables L_B and L_E with density functions p_B and p_E, respectively. Then, with $i^2 = -1$,

$$B_n(x) = \mathbb{E}\left[\left(iL_B + x - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(it + x - \frac{1}{2}\right)^n p_B(t)\mathrm{d}t, \quad [5, eq.\ 2.14]$$

$$E_n(x) = \mathbb{E}\left[\left(iL_E + x - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(it + x - \frac{1}{2}\right)^n p_E(t)\mathrm{d}t. \quad [9, eq.\ 2.3]$$

3rd Reason

- The generalized Motzkin numbers:
  1. Motzkin numbers: $M_{0,0} = 1$, $M_{n,k=0}$ if $k > n$ or $n < 0$, and

$$M_{n+1,k} = M_{n,k-1} + M_{n,k} + M_{n,k+1}.$$

$$\begin{array}{ccccc} M_{n,k-1} & & M_{n,k} & & M_{n,k+1} \\ & \searrow & \downarrow & \swarrow & \\ & & M_{n+1,k} & & \end{array}$$

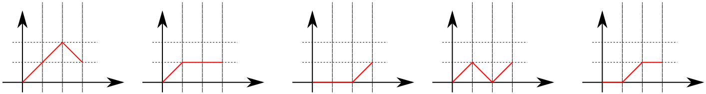
$$M_{n,k} = \#\text{ of } \underline{\textit{paths}} \text{ from } (0,0) \text{ to } (n,k).$$

Path: Lattice path on $\mathbb{N}^2$ $(0 \in \mathbb{N})$ and only three types are considered:

$$\begin{cases} \alpha_k : (j,k) \to (j+1,k+1) & \text{diagonally up } \nearrow \\ \beta_k : (j,k) \to (j+1,k); & \text{horizontal } \to \\ \gamma_k : (j,k) \to (j+1,k-1) & \text{diagonally down } \searrow \end{cases}$$

Example.

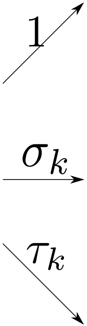
$$\begin{matrix} M_{0,k} \\ M_{1,k} \\ M_{2,k} \\ M_{3,k} \end{matrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 4 & 5 & 3 & 1 \end{pmatrix} \Rightarrow M_{3,1} = 5$$



3rd Reason

- The generalized Motzkin numbers:
  2. generalized Motzkin numbers:

$$M_{n+1,k} = M_{n,k-1} + \sigma_k M_{n,k} + \tau_{k+1} M_{n,k+1}$$



3rd Reason

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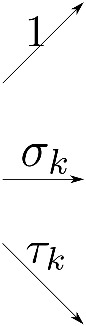


$M_{n,k}$ = sum of weighted lattice paths from $(0,0)$ to (n,k).

$$\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - \sigma_0 z - \cfrac{\tau_1 z^2}{1 - \sigma_1 z - \cfrac{\tau_2 z^2}{\cdots}}}$$

Let X be an arbitrary random variable, with moments m_n and monic orthogonal polynomials $P_n(y)$ satisfying the recurrence

$$P_{n+1}(y) = (y - s_n) P_n(y) - t_n P_{n-1}(y).$$

Then, we have

$$\sum_{n=0}^{\infty} m_n z^n = \cfrac{m_0}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{1 - s_2 z - \cdots}}}.$$

Continued Fractions

By letting $(\sigma_k, \tau_k) = (s_k, t_k)$. If further assuming $m_0 = 1$, we have

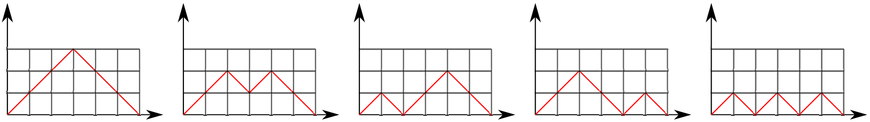
$$M_{n,0} = m_n = \mathbb{E}[X^n].$$

Example. The monic orthogonal polynomials with respect to E_n, $Q_n(y)$, satisfy

$$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y).$$

Euler numbers E_n are given by the weighted lattice paths $(1, 0, -k^2) \Rightarrow$ horizontal paths are eliminated. Therefore, E_n counts the weighted *Dyck paths*, related to Catalan numbers C_n.
$\underline{n = 6}$:

$$C_3 := \frac{1}{4}\binom{6}{3} = 5$$



Then, by noting that each diagonally down path from (j, k) to $(j + 1, k - 1)$ has weight $-k^2$, we have

$$-61 = E_6 = (-1)^3 \left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right).$$

1st Task

- Find the orthogonal polynomials with respect to $B_n(x)$, denoted by $\varrho_n(y)$;
- and the orthogonal polynomials with respect to $E_n(x)$, denoted by $\Omega_n(y)$.

Namely, for $0 \leq r < n$,

$$y^r \varrho_n(y)\Big|_{y^k = B_k(x)} = 0 = y^r \Omega_n(y)\Big|_{y^k = E_k(x)}.$$

NOTE:

$$B_n(x) = \mathbb{E}\left[\left(iL_B + x - \frac{1}{2}\right)^n\right] \quad \text{and} \quad B_n = B_n(0) = \mathbb{E}\left[\left(iL_B - \frac{1}{2}\right)^n\right],$$

and the monic orthogonal polynomials with respect to the B_n, denoted by $R_n(y)$, are given by

$$R_{n+1}(y) = \left(y + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y).$$

Let c be a constant.

$$\begin{aligned} X &\sim P_n(y) \\ X + c &\sim\ ? \\ cX &\sim\ ? \end{aligned}$$

$$E_n = 2^n E_n(1/2)$$

1st Task

Lemma. [L. Jiu and D. Shi]

random variable	moments	monic orthogoal polynomial
X	m_n	$P_n(y): \quad P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y)$
$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$\bar{P}_n(y): \quad \bar{P}_{n+1}(y) = (y - s_n - c)\bar{P}_n(y) - t_n \bar{P}_{n-1}(y)$
CX	$C^n m_n$	$\tilde{P}_n(y): \quad \tilde{P}_{n+1}(y) = (y - Cs_n)\tilde{P}_n(y) - C^2 t_n \tilde{P}_{n-1}(y)$

Proof.

$$\bar{P}_n(y) := P_n(y - c) \quad \text{and} \quad \tilde{P}_n(y) := C^n P_n(y/C).$$

□

Theorem. [L. Jiu and D. Shi]

B_n	$R_{n+1}(y) = \left(y + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y)$
$B_n(x)$	$\varrho_{n+1}(y) = \left(y - x + \frac{1}{2}\right) \varrho_n(y) + \frac{n^4}{4(2n+1)(2n-1)} \varrho_{n-1}(y)$
E_n	$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y)$
$E_n(x)$	$\Omega_{n+1}(y) = \left(y - x + \frac{1}{2}\right) \Omega_n(y) + \frac{n^2}{4} \Omega_{n-1}(y)$

2nd Task

- Find the orthogonal polynomials with respect to $B_n^{(p)}(x)$, denoted by $\varrho_n^{(p)}(y)$;
- and the orthogonal polynomials with respect to $E_n^{(p)}(x)$, denoted by $\Omega_n^{(p)}(y)$.

Recall that

- Bernoulli polynomial $B_n(x)$:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!};$$

- Euler polynomial $E_n(x)$:

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!};$$

- Bernoulli polynomial of order p $B_n^{(p)}(x)$:

$$\left(\frac{t}{e^t - 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!};$$

- Euler polynomial of order p $E_n^{(p)}(x)$:

$$\left(\frac{2}{e^z + 1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x) \frac{z^n}{n!};$$

- $$B_n(x) = \mathbb{E}\left[\left(iL_B + x - \frac{1}{2}\right)^n\right]$$

- $$E_n(x) = \mathbb{E}\left[\left(iL_E + x - \frac{1}{2}\right)^n\right]$$

$$B_n^{(p)}(x) = \mathbb{E}\,[?] \quad E_n^{(p)}(x) = \mathbb{E}\,[?]$$

Sum of random variables

Let X and Y be two independent variables, with moments m_n and m'_n, respectively. In addition, define the moment generating functions:

$$F(t) := \mathbb{E}\left[e^{tX}\right] = \sum_{n=0}^{\infty} m_n \frac{t^n}{n!} \quad \text{and} \quad G(t) := \mathbb{E}\left[e^{tY}\right] = \sum_{n=0}^{\infty} m'_n \frac{t^n}{n!}$$

Then,

$$\mathbb{E}\left[(X+Y)^n\right] = \sum_{k=0}^{n} \binom{n}{k} m_k m'_{n-k}$$

and

$$\mathbb{E}\left[e^{t(X+Y)}\right] = F(t)G(t).$$

Consider a sequence of independent and identically distributed (i. i. d.) random variables $\left(L_{E_j}\right)_{j=1}^{p}$ with each $L_{E_j} \sim L_E(\operatorname{sech}(t))$. Then $E_n^{(p)}(x)$ is the nth moment of a certain random variable:

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + \sum_{j=1}^{p} iL_{E_j} - \frac{p}{2}\right)^n\right].$$

Similarly, given an i. i. d. $\left(L_{B_j}\right)_{j=1}^{p}$ with $L_{B_j} \sim L_B(\pi \operatorname{sech}^2(\pi t)/2)$,

$$B_n^{(p)}(x) = \mathbb{E}\left[\left(x + \sum_{j=1}^{p} iL_{B_j} - \frac{p}{2}\right)^n\right].$$

Hankel Determinant

Given a sequence $\mathbf{a} = (a_n)_{n=0}^{\infty}$, the *nth Hankel determinant* of $\mathbf{a}$ is defined by

$$\Delta_n(\mathbf{a}) := \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}.$$

Recall that given a sequence of numbers/moments $\mathbf{m} := (m_n)_{n=0}^{\infty}$, let $P_n(y)$ be the monic orthogonal polynomials with respect to m_n. Namely, for all $0 \leq r < n$

$$y^r P_n(y)\Big|_{y^k = m_k} = 0.$$

Suppose P_n satisfies the three-term recurrence:

$$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y).$$

Theorem. (1) If $m_{2k+1} = 0$ for all $k \in \mathbb{N}$, then, $s_n = 0$;
Recall The monic orthogonal polynomials with respect to E_n, $Q_n(y)$, satisfy

$$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y).$$

And

$$E_{2k+1} = 0. \qquad \left(\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2e^t}{e^{2t}+1} = \operatorname{sech}(t). \right)$$

Hankel Determinant

$$\Delta_n(\mathbf{a}) := \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}.$$

$$\mathbf{m} := (m_n)_{n=0}^{\infty} \longleftrightarrow P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y).$$

Theorem. (2) Define $\mathbf{a}(c) = (a_n(c))_{n=0}^{\infty}$ by

$$a_n(c) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} c^k.$$

Then,

$$\Delta_n(\mathbf{a}(c)) = \Delta_n(\mathbf{a}).$$

(3) [11, Thm. 11, p. 20]

$$\Delta_n(\mathbf{m}) = m_0^{n+1}(-t_1)^n(-t_2)^{n-1}\cdots(-t_{n-1})^2(-t_n),$$

which simpliy implies that

$$-t_n = \frac{\Delta_n(\mathbf{m})\Delta_{n-2}(\mathbf{m})}{[\Delta_{n-1}(\mathbf{m})]^2}.$$

Recall

X	m_n	$P_n(y)$: $P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y)$
$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$\bar{P}_n(y)$: $\bar{P}_{n+1}(y) = (y - s_n - c)\bar{P}_n(y) - t_n \bar{P}_{n-1}(y)$

2nd Task

Theorem. (4)

$$P_n(y) = \frac{1}{\Delta_{n-1}(\mathbf{m})} \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & y & y^2 & \cdots & y^n \end{pmatrix}$$

- Find the orthogonal polynomials with respect to $B_n^{(p)}(x)$, denoted by $\varrho_n^{(p)}(y)$;
- and the orthogonal polynomials with respect to $E_n^{(p)}(x)$, denoted by $\Omega_n^{(p)}(y)$.

By the generating functions

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^z+1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!},$$

we have

$$B_{2k+1}^{(p)}(p/2) = 0 = E_{2k+1}^{(p)}(p/2).$$

Then, with the lemma for shifted and scaled random variables, we see

$$\varrho_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\varrho_n^{(p)}(y) + b_n^{(p)}\varrho_{n-1}^{(p)}(y)$$

$$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + e_n^{(p)}\Omega_n^{(p)}(y)$$

Conjecture on $b_n^{(p)}$

The first several terms of $b_n^{(p)}$ is given in the following table

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

- The first column has formula

$$\frac{n^4}{4(2n+1)(2n-1)}$$

$$R_{n+1}(y) = \left(y - x + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y)$$

- The first row is $p/12$
- The second row is $(5p+3)/30$

Conjecture on $b_n^{(p)}$

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
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Conjecture. [K. Dilcher]

$$b_3^{(p)} = \frac{175p^2 + 315p + 158}{140(2p+3)};$$

$$b_4^{(p)} = \frac{6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230}{21(5p+3)(175p^2 + 315p + 158)};$$

$$b_5^{(p)} = 25(5p+3)(471625p^6 + 3678675p^5 + 12324235p^4 + 22096305p^3 + 22009540p^2 + 11549748p + 2519472) \Big/ (132(175p^2 + 315p + 158)(6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230)).$$

Difficulties

- Guessing polynomials (rational functions)

$$
\begin{aligned}
1+2+\cdots+n &= \frac{n(n+1)}{2} \\
1^2+2^2+\cdots+n^2 &= \frac{n(n+1)(2n+1)}{6} \\
&\cdots \\
1^k+2^k+\cdots+n^k &= \sum_{j=0}^{k+1} a_j n^j \text{ (a polynomial in variable } n \text{ of degree } k+1) \\
&= \frac{1}{k+1}\left[B_{k+1}(n+1) - B_{k+1}\right]
\end{aligned}
$$

-

random variable	moments	monic orthogoal polynomial
X	m_n	$P_n(y): \quad P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y)$
$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$\bar{P}_n(y): \quad \bar{P}_{n+1}(y) = (y - s_n - c)\bar{P}_n(y) - t_n \bar{P}_{n-1}(y)$
CX	$C^n m_n$	$\tilde{P}_n(y): \quad \tilde{P}_{n+1}(y) = (y - Cs_n)\tilde{P}_n(y) - C^2 t_n \tilde{P}_{n-1}(y)$
$X + Y$	Convolution	???

How about $e_n^{(p)}$?

The first several terms of $e_n^{(p)}$ is given in the following table

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{4}$	1	$\frac{9}{4}$	4	$\frac{25}{4}$
$n=2$	$\frac{1}{2}$	$\frac{3}{2}$	3	5	$\frac{15}{2}$
$n=3$	1	2	$\frac{15}{4}$	6	$\frac{35}{4}$
$n=4$	1	$\frac{5}{2}$	$\frac{9}{2}$	7	10
$n=5$	$\frac{5}{4}$	3	$\frac{21}{4}$	8	$\frac{45}{4}$

- $$e_n^{(p)} = \frac{n\,(n+p-1)}{4}$$
- **Theorem**. [L. Jiu and D. Shi]

$$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + \frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y).$$

Meixner-Pollaczek polynomials

The *Meixner-Pollaczek polynomials* are defined by

$$P_n^{(\lambda)}(y;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(\left.\begin{matrix}-n, \lambda+iy \\ 2\lambda\end{matrix}\right| 1-e^{-2i\phi}\right),$$

where $(x)_n := x(x+1)(x+2)\cdots(x+n-1)$ is the Pochhammer symbol and ${}_2F_1$ is the hypergeometric function

$${}_pF_q\left(\left.\begin{matrix}a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right| t\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{t^n}{n!}.$$

Recurrence.

$$(n+1)P_{n+1}^{(\lambda)}(y;\phi) = 2(y\sin\phi+(n+\lambda)\cos\phi)P_n^{(\lambda)}(y;\phi) - (n+2\lambda-1)P_{n-1}^{(\lambda)}(y;\phi).$$

KEY.

$$P_n^{(\lambda+\mu)}(y_1+y_2,\phi) = \sum_{k=0}^{n} P_k^{(\lambda)}(y_1,\phi)\, P_{n-k}^{(\mu)}(y_2,\phi).$$

Theorem. [L. Jiu and D. Shi]

$$\Omega_n^{(p)}(y) = \frac{i^n n!}{2^n} P_n^{(\frac{p}{2})}\left(-i\left(y-x+\frac{p}{2}\right);\frac{\pi}{2}\right).$$

Fact. Euler numbers E_n have monic orthogonal polynomials $Q_n(y)$:

$$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y),$$

$$Q_n(y) := i^n n! P_n^{(\frac{1}{2})}\left(\frac{-iy}{2};\frac{\pi}{2}\right).$$

Continuous Hahn Polynomials

The *Continuous Hahn* polynomial is defined by

$$p_n(x;a,b,c,d) = i^n \frac{(a+c)_n(a+d)_n}{n!} \, {}_3F_2\left(\left. \begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} \right| 1 \right).$$

Fact. Recall the orthogonal polynomials with respect to $B_n(x)$, denoted by $\varrho_n(y)$. Then,

$$\varrho_n(y) = \frac{n!}{(n+1)_n} p_n\left(y; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

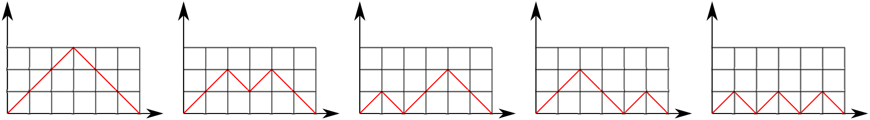
The key property for Meixner-Pollaczek polynomials

$$P_n^{(\lambda+\mu)}(y_1+y_2,\phi) = \sum_{k=0}^{n} P_k^{(\lambda)}(y_1,\phi)\, P_{n-k}^{(\mu)}(y_2,\phi)$$

does not hold for continuous Hahn polynomials.

What's Next?

Recall that

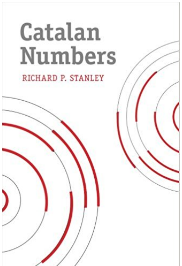


$$-61 = E_6 = (-1)^3\left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right).$$

Aim:

$$E_{2n} = \sum_{j=1}^{C_n} f\left(-1^2, -2^2, \ldots, -n^2\right)?$$

Try: recall $C_1 = 1$, $C_2 = 2$, $C_3 = 5$ and $C_4 = 14$. "Chalk Work"



What's Next?

$$R_{n+1}(y) = \left(y + \frac{1}{2}\right) R_n(y) + \frac{n^4}{4(2n+1)(2n-1)} R_{n-1}(y)$$

n	0	1	2	3	4
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$

For any $0 \leq r < n$

$$y^r R_n(y)\Big|_{y^k = B_k} = 0 \Rightarrow R_n(y)\Big|_{y^k = B_k} = 0$$

- $R_0 = 1$;
- $R_1 = y + \frac{1}{2}$: $R_1(y)\Big|_{y^k = B_k} = B_1 + \frac{1}{2} B_0 = -\frac{1}{2} + \frac{1}{2} = 0$;
- $R_2 = \left(y + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) + \frac{1}{12} = y^2 + y + \frac{1}{3}$:
  $R_2(y)\Big|_{y^k = B_k} = B_2 + B_1 + \frac{1}{3} = \frac{1}{6} - \frac{1}{2} + \frac{1}{3} = 0$;
- $R_3 = \left(y + \frac{1}{2}\right)\left(y^2 + y + \frac{1}{3}\right) + \frac{4}{15}\left(y + \frac{1}{2}\right) = y^3 + \frac{3}{2} y^2 + \frac{11}{10} y + \frac{3}{10}$:
  $R_3(y)\Big|_{y^k = B_k} = B_3 + \frac{3}{2} B_2 + \frac{11}{10} B_1 + \frac{3}{10} = 0$;

What's Next?

$$B_n(x+1) - B_n(x) = nx^{n-1} \Leftrightarrow B_n(x+1) - B_n(x) - nx^{n-1} = 0$$

$$P(n;y) = (y+x+1)^n - (y+x)^n - nx^{n-1} \; (= \bar{P}_{n-1}(y))$$

$$P(n;y)\Big|_{y^k = B_k} = 0.$$

Recall that $\deg R_n = n$, and $R_n(y)\Big|_{y^k=B_k}$ for $n > 0$.

Proposition.

$$P(n;y) = \sum_{k=1}^{n-1} \alpha_{n,k} R_k(y).$$

For some constants $\alpha_{n,k}$ that are independent of y.
Proof. By induction on the degree of P.

What's Next?

Exmaple.

$$P(n;y) = (y+x+1)^n - (y+x)^n - nx^{n-1}$$
$$R_1 = y + \frac{1}{2}$$
$$R_2 = y^2 + y + \frac{1}{3}$$
$$R_3 = y^3 + \frac{3}{2}y^2 + \frac{11}{10}y + \frac{3}{10}$$

- $P(2;y) = (y+x+1)^2 - (y+x)^2 - 2x = 2y + 1 = 2R_1$;
- $P(3;y) = 3y^2 + (3+6x)y + 3x + 1 = 3\left(y^2 + y + \frac{1}{3}\right) + 6xy + 3x$
  $= 3R_2 + 6xR_1$;
- $P(4;y) = 4R_3 + 12xR_2 + (12x^2 - \frac{2}{5})R_1$.

What's Next?

- q-analogue?
- hypergeometric Bernoulli numbers:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} = \frac{1}{{}_1F_1\left(\left.\begin{matrix}1\\2\end{matrix}\right| t\right)}.$$

$$\frac{e^t - 1}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$$

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$$\frac{e^t - 1}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{t^n}{n!}$$

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$$\frac{e^t - 1}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(1)_n}{(2)_n} \cdot \frac{t^n}{n!}.$$

Define

$$\frac{e^{xt}}{{}_1F_1\left(\begin{matrix}a\\a+b\end{matrix}\middle| t\right)} = \sum_{n=0}^{\infty} B_n^{(a,b)}(x) \frac{t^n}{n!}.$$

End

Thank you!

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