Zeros and irreducibility of some classes of special polynomials

Karl Dilcher

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Part I: Chebyshev-like polynomials

Pafnutiy L'vovich Chebyshev
1821 – 1894

Joint work with

Kenneth B. Stolarsky
University of Illinois, Urbana-Champaign

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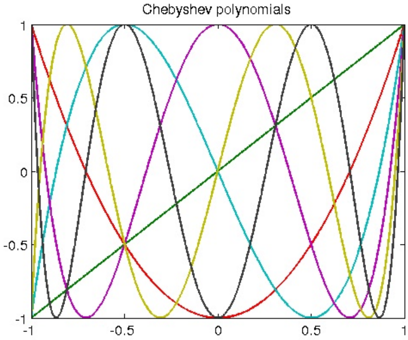

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1	x
2	$x^2 - 1$
3	$x^3 - 3x$
4	$x^4 - 7x^2 + 1$
5	$x^5 - 15x^3 + 5x$
6	$x^6 - 31x^4 + 17x^2 - 1$
7	$x^7 - 63x^5 + 49x^3 - 7x$
8	$x^8 - 127x^6 + 129x^4 - 31x^2 + 1$
9	$x^9 - 255x^7 + 321x^5 - 111x^3 + 9x$
10	$x^{10} - 511x^8 + 769x^6 - 351x^4 + 49x^2 - 1$
11	$x^{11} - 1023x^9 + 1793x^7 - 1023x^5 + 209x^3 - 11x$
12	$x^{12} - 2047x^{10} + 4097x^8 - 2815x^6 + 769x^4 - 71x^2 + 1$

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Compare with

$$T_n(x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (x^2 - 1)^{k} x^{n-2k},$$

from which (2) is derived, by way of (1).

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Easy to see:

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We can prove a partial result:

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Sketch of Proof: Using the explicit expansion

$$V_n(x) = x^n - \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^r\left(\sum_{k=r+1}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{k-1}{r}\right)x^{n-2-2r},$$

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it can be shown that the polynomials in (a) and (b) are 2-Eisenstein.

(No other $V_{2k}(x)$ or $\frac{1}{x}V_{2k+1}(x)$ is Eisenstein).

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n	r_n	n	r_n
1	0	11	31.956928
2	1	12	45.221645
3	1.7320508	13	63.974591
4	2.6180339	14	90.490325
5	3.8286956	15	127.98534
6	5.5174860	16	181.00828
7	7.8875983	17	255.99169
8	11.223990	18	362.03245
9	15.929112	19	511.99536
10	22.571929	20	724.07389

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4	2.6180339	2.8284271	14	90.490325	90.509667
5	3.8286956	4	15	127.98534	128
6	5.5174860	5.6568542	16	181.00828	181.01933
7	7.8875983	8	17	255.99169	256
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Let $n \geq 2$, and $\pm r_n$ be the largest zeros in absolute value of $V_n(x)$. Then

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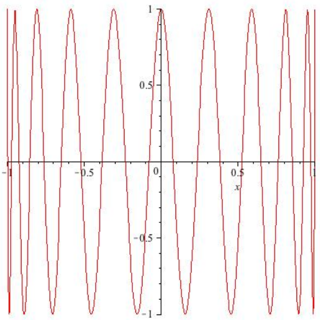

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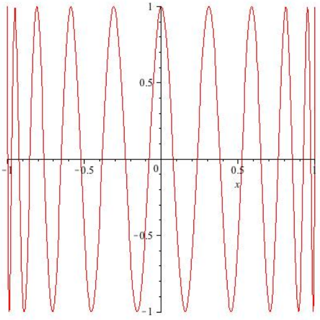

(b): Evaluate $V_n(x)$ at the two boundary points of the interval.

$T_{20}(x)$

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The Chebyshev polynomials $T_n(x)$ satisfy the (2×2 Hankel determinant) identity

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We'll see: These polynomials have some interesting properties.

n	$W_n(x)$
0	1
1	x^2+1
2	$2x^4+x^2+1$
3	$4x^6+x^4+x^2+1$
4	$8x^8+x^4+x^2+1$
5	$16x^{10}-4x^8+x^6+x^4+x^2+1$
6	$32x^{12}-16x^{10}+2x^8+x^6+x^4+x^2+1$
7	$64x^{14}-48x^{12}+8x^{10}+x^8+x^6+x^4+x^2+1$
8	$128x^{16}-128x^{14}+32x^{12}+x^8+x^6+x^4+x^2+1$
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Do we get anything sensible if we cut the $W_n(x)$ into two halves?

Define the lower and upper parts, respectively, of $W_n(x)$ by

$$W_n^{\ell}(x) := \sum_{j=0}^{\lfloor\frac{n+1}{2}\rfloor} x^{2j},$$

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Easy to establish generating functions for both, and with these we get

$$W_n^u(x) = 2 \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} U_{n-2-2k}(x)$$

where the $U_n(x)$ are the Chebyshev polynomials of the second kind, which can be defined by the generating function

$$\frac{1}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n.$$

Using known identities:

$$W_{2k}^{u}(x) = \frac{1 - T_{2k}(x)}{1 - x^2} = 2U_{k-1}(x)^2,$$
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This, together with the definition of the $W_n^{\ell}(z)$, gives

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For all $n \geq 1$, the zeros

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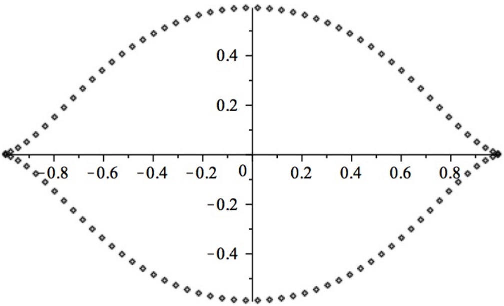

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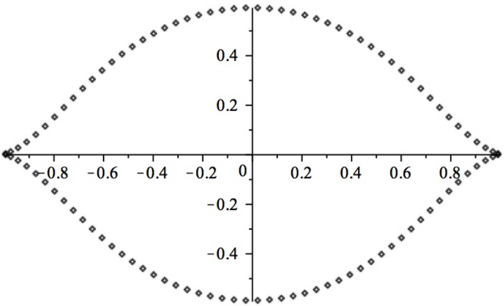

Do they lie on (or near) an identifiable curve?

Proposition

The zeros of $W_n(z)$, as $n \to \infty$, lie arbitrarily close to the curve

$$3r^8 - 8r^6\cos(2\theta) + 6r^4 - 1 = 0, \qquad z = re^{i\theta},\ 0 \le \theta \le 2\pi. \quad (4)$$

Furthermore, they all lie outside the closed region defined by this curve.

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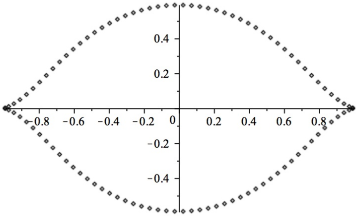

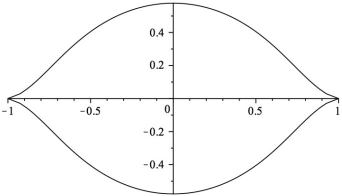

Figure: The zeros of $W_{50}(z)$ and the curve (4).

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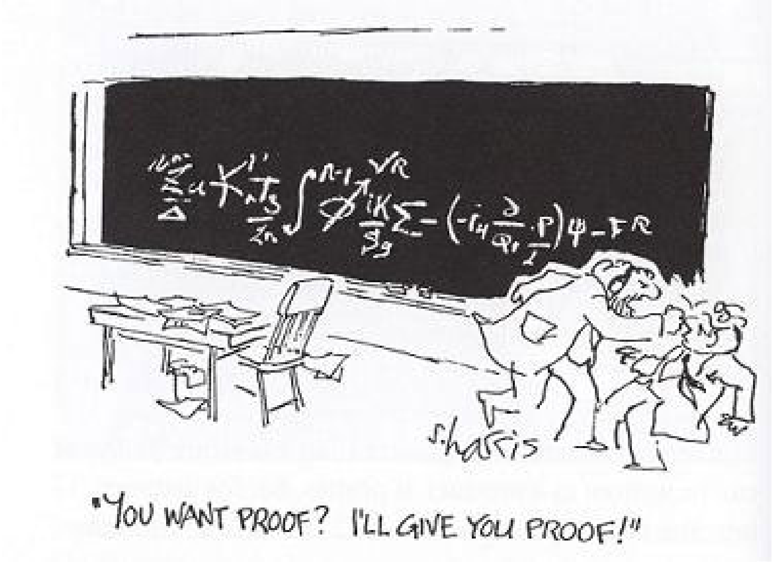

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- Concentrate on the larger of the two summands.
- A chain of tricky estimates.

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Let $Lp(x)$, $Up(x)$ be the lower and upper sections of an even-degree polynomial $p(x)$.

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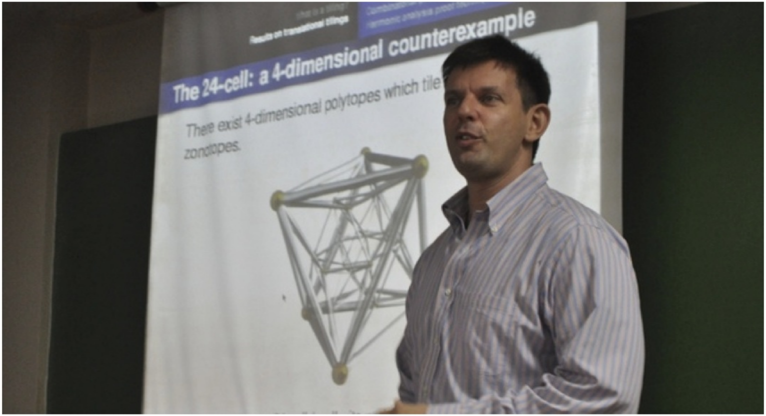
(ii) The polynomials can be given explicitly and are also related to Chebyshev polynomials.

Part II:

Zeros and irreducibility of gcd-polynomials

Joint work with



Sinai Robins

University of São Paulo, Brazil

1. Introduction

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Deep connections with the distribution of primes.

2. Ramanujan polynomials:

$$R_{2k+1}(z) := \sum_{j=0}^{k+1} \left(\frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} \right) z^{2j},$$

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Later extended by other authors to similar polynomials (Lalín & Smyth, 2013; Berndt & Straub, 2017).

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with $((x))$ denoting the “sawtooth function"

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In an effort to prove this, we were led to studying the following class of polynomials.

2. GCD Polynomials

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For $n = p - 1$ (p a prime), these are cyclotomic polynomials; hence irreducible.

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Idea of proof: Consider

$$g_n^{(k)}(e^{2\pi i x})$$

and show it has n *real* zeros for $0 < x < 1$.

3. Zeros; proof of the Theorem

Since $\gcd(j,n)=\gcd(n-j,n)$ for $0\le j\le n$, the $g_n^{(k)}(z)$ are *self-inversive* (or *reciprocal*):

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Theorem (L. Tóth, 2011)

For all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$S^{(1)}(m,n) = \sum_{d \mid \gcd(m,n)} d\varphi(\tfrac{n}{d}).$$

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Can this be extended to general $k \geq 1$?

4. Jordan's totient function

We need a generalization of Euler's φ-function.

Definition

Jordan's totient function is defined by

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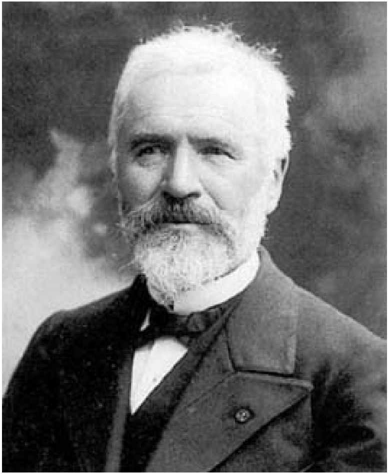
This equivalence was first established by Camille Jordan in 1870.

Clearly, $J_1(n) = \varphi(n)$.



Camille Jordan
(1838–1922)

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This can be extended:

Proposition

For all $k, n \in \mathbb{N}$ *and all* $m \in \mathbb{Z}$ *we have*

$$S^{(k)}(m,n) = \sum_{d \mid \gcd(m,n)} d J_k\left(\tfrac{n}{d}\right).$$

In particular, $S^{(k)}(m,n)$ *is always a positive integer.*

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Compare with Tóth's result:

$$S^{(1)}(m,n) = \sum_{d \mid \gcd(m,n)} d\varphi(\tfrac{n}{d}).$$

Proof of Proposition. Using

$$\gcd(j,n)^k = \sum_{d \mid \gcd(j,n)} J_k(d),$$

we have

$$\begin{aligned}
S^{(k)}(m,n) &= \sum_{j=1}^{n} \sum_{\ell \mid \gcd(n,j)} J_k(\ell) e^{2\pi ijm/n} \\
&= \sum_{\ell \mid n} J_k(\ell) \sum_{j=1}^{n/\ell} e^{2\pi ijm/(n/\ell)}.
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Hence, setting $d = n/\ell$, we get the desired identity.

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This was published by K. Alladi (1975) when he was 19 years old, and with a different goal in mind.

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However, it appears that this is the *only* factor. In fact:

Theorem

For $\alpha, k \in \mathbb{N}$ and odd primes p,

$$g_{2^\alpha}^{(k)}(z) \quad \textit{and} \quad \frac{g_{p^\alpha}^{(k)}(z)}{z+1}$$

are irreducible over $\mathbb{Q}$.

Proof. (Sketch).

Part 1: We begin with the smallest cases:

$$g_2^{(k)}(z) = 2^k + z + 2^k z^2, \qquad \frac{1}{z+1} g_3^{(k)}(z) = 3^k + (1-3^k)z + 3^k z^2.$$

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But none of the polynomials above are of this form.
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For the remaining cases, let $p \geq 2$ be any prime, and $\alpha, k \in \mathbb{N}$. Set

$$\overline{g}_n^{(k)}(z) = \begin{cases} g_n^{(k)}(z) & \text{when } n \text{ is even,} \\ \frac{1}{z+1} g_n^{(k)}(z) & \text{when } n \text{ is odd.} \end{cases}$$

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So we can write, for any $n \geq 4$,

$$\begin{aligned}\overline{g}_n^{(k)}(z) &= (a_1 + b_1 z + \dots)(a_2 + b_2 z + \dots)\dots(a_r + b_r z + \dots)\\ &= a_1 a_2 \dots a_r + a_1 a_2 \dots a_r \left(\sum_{j=1}^{r} \frac{b_j}{a_j}\right) z + \dots\end{aligned}$$

On the other hand, it is clear from the definition that

$$\overline{g}_{p^\alpha}^{(k)}(z) = \begin{cases} p^{\alpha k} + (1 - p^{\alpha k})z + \dots & \text{when } p \geq 3, \\ p^{\alpha k} + z + \dots & \text{when } p = 2. \end{cases}$$

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Equating coefficients, we therefore have

$$a_1 a_2 \dots a_r = p^{\alpha k}, \tag{6}$$

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- By (6): the a_j can only be powers of p;
- by (7): at least one of them has to be 1

(otherwise p would divide LHS of (7) — contradiction.)

This means: at least one of the r irreducible factors (which are self-inversive) is monic, with all its zeros on the unit circle.

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Leopold Kronecker 1823 – 1891

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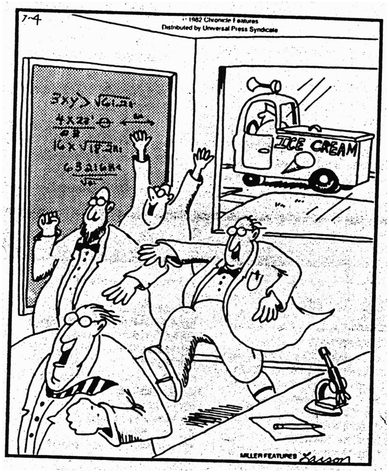
Proof requires a detailed analysis using resultants of polynomials.

We skip this.

Thank you



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We denote the resultant of f and g by

$$\mathrm{Res}(f, g)$$

if there is no ambiguity as to the variable z.

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$$\mathrm{Res}(f, g) = a_m^n b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j),$$

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The first identity shows that $\mathrm{Res}(f, g) = 0$
iff f and g have a factor in common.

Important for us:

Lemma (Apostol (1970))

For $m > n > 1$ we have

$$\mathrm{Res}(\Phi_m(z), \Phi_n(z)) = \begin{cases} p^{\varphi(n)} & \text{if } \frac{m}{n} \text{ is a power of a prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

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Let p be any prime and α, k be positive integers. Then

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unless $n = 2^j d$ for some nonzero j and $d > 1$ where $d \mid p^\alpha + 1$ (j may be positive or negative).

(b) On the other hand,

$$
\begin{aligned}
g_{p^\alpha}^{(k)}(z) &\equiv (z+\cdots+z^{p-1})+(z^{p+1}+\cdots+z^{2p-1})\\
&\quad+\cdots+(z^{p\alpha-p+1}+\cdots+z^{p\alpha-1}) \pmod p\\
&= z\left(1+z+\cdots+z^{p-2}\right)\left(1+z^p+\cdots+z^{(p^{\alpha-1}-1)p}\right)\\
&= z\cdot\frac{z^{p-1}-1}{z-1}\cdot\frac{z^{p^\alpha}-1}{z^p-1}\\
&= z\prod_{\substack{d\mid p-1\\ d\neq 1}}\Phi_d(z)\prod_{j=2}^{\alpha}\Phi_{p^j}(z).
\end{aligned}
$$

By properties of resultants,

$$\mathrm{Res}(z, \Phi_n(z)) = 1 \quad \text{for} \quad n \geq 3,$$

and so

$$\mathrm{Res}(g_{p^\alpha}^{(k)}(z), \Phi_n(z)) \equiv \pm \prod_{\substack{d|p-1 \\ d\neq 1}} \mathrm{Res}(\Phi_d(z), \Phi_n(z)) \times \prod_{j=2}^{\alpha} \mathrm{Res}(\Phi_{p^j}(z), \Phi_n(z)) \pmod{p}.$$

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The above congruences (mod 2) and (mod p) fail simultaneously only if

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Impossible for an odd prime p since $\ell \geq 1$ and $p \nmid d_1$.

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This completes the proof of the resultant lemma, and thus of the irreducibility theorem.

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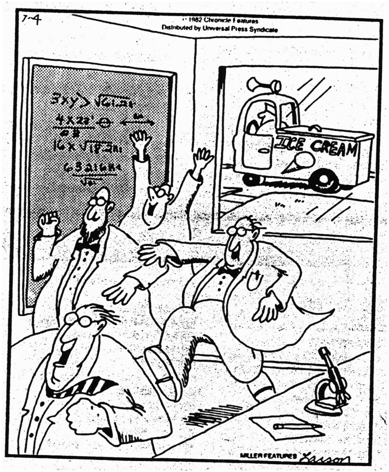
2. Our results give a large supply of algebraic numbers on the unit circle that are not roots of unity.

Thank you



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