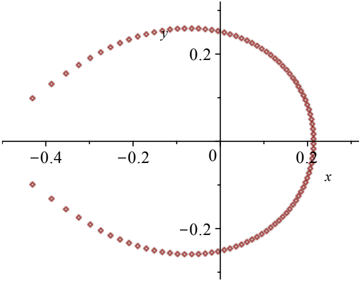
Some polynomial and geometric Diophantine equations

Karl Dilcher

Number Theory Seminar, March 11, 2019

Part I

Polynomial Diophantine Equations



Joint work with

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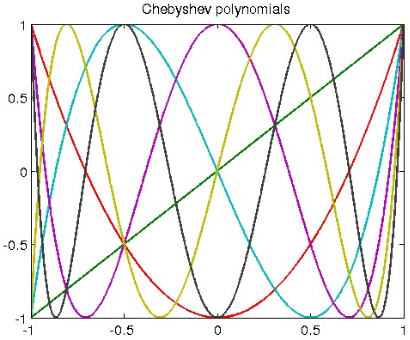

Similarly, Chebyshev polynomials of the second kind, $U_n(x)$, can be defined by $U_0(x) = 1$, $U_1(x) = 2x$, and

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$T_n(x)$ and $U_n(x)$ can also be defined as solutions of the polynomial Pell equation

$$T_n(x)^2 - (x^2 - 1)U_{n-1}(x)^2 = 1$$

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Here we'll consider a variant of this equation.

$\mathbb{Q}[x]$ is a Euclidean domain, so for given coprime $f, g \in \mathbb{Q}[x]$ there are $P, Q \in \mathbb{Q}[x]$ with

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Purpose: To study the sequences $P_n(x), Q_n(x)$.

n	$P_n(x)$
0	-1
1	$2x+3$
2	$-6x^2-15x-10$
3	$20x^3+70x^2+84x+35$
4	$-70x^4-315x^3-540x^2-420x-126$

n	$Q_n(x)$
0	1
1	$-2x+1$
2	$6x^2-3x+1$
3	$-20x^3+10x^2-4x+1$
4	$70x^4-35x^3+15x^2-5x+1$

Table 1: $P_n(x)$ and $Q_n(x)$ for $0 \leq n \leq 4$.

2. Basic Properties

Proposition

For $n \geq 0$ we have $\deg P_n = \deg Q_n$, and

$$P_n(x) = (-1)^{n+1} Q_n(-1-x), \quad Q_n(x) = (-1)^{n+1} P_n(-1-x).$$

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Corollary

$$P_n(-1) = (-1)^{n+1}, \quad P_n(-\tfrac{1}{2}) = (-1)^{n+1} 2^n,$$
$$Q_n(-\tfrac{1}{2}) = 2^n, \quad Q_n(0) = 1.$$

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Proof: In $P_n(x)x^{n+1} + Q_n(x)(x+1)^{n+1} = 1$, set $x = 0, -1$, and $-1/2$.

Explicit formulas:

Proposition

For any $n \geq 0$ *we have* $\deg Q_n = degP_n = n$*, and*

$$Q_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n+i}{i} x^i, \qquad (1)$$

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$$P_n(x) = (-1)^{n+1}(2n+1)\binom{2n}{n} \sum_{i=0}^{n} \frac{1}{n+i+1} \binom{n}{i} x^i. \tag{2}$$

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Idea of Proof: For (1):

- Differentiate $P_n(x)x^{n+1} + Q_n(x)(x+1)^{n+1} = 1$;
- make some divisibility arguments;
- use induction.

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- make some divisibility arguments;
- use induction.

For (2): Use (1) and $P_n(x) = (-1)^{n+1} Q_n(-1-x)$.

Proposition

For $1 \leq k \leq n+1$ *we have*

$$(x+1)Q_n^{(k)}(x)+(n+k)Q_n^{(k-1)}(x) = (-1)^n\frac{(2n+1)!}{n!}\frac{x^{n-k+1}}{(n-k+1)!},$$

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Homogeneous ODE:

Corollary

For $n \geq 0$ *we have*

$$x(x+1)Q_n''(x)+(2x-n)Q_n'(x)-n(n+1)Q_n(x)=0.$$

Recurrence relation:

Proposition

$Q_0(x) = 1$ and $Q_1(x) = -2x + 1$, and for $n \geq 2$,

$$n(x+1)Q_n(x) = -\big(2(2n-1)x^2 + 2(2n-1)x - n\big)Q_{n-1}(x) \\ + 2(2n-1)xQ_{n-2}(x).$$

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Consequence: Generating function.

Corollary

$$\frac{1 + xt + (1+2x)\sqrt{1+4xt}}{2(1+x-t)(1+4xt)} = \sum_{n=0}^{\infty} Q_n(x)t^n.$$

3. Resultants

Suppose we have the two polynomials

$$f(x) = a_0 x^\mu + \cdots + a_{\mu-1} x + a_\mu = a_0(x - \alpha_1)\cdots(x - \alpha_\mu),$$
$$g(x) = b_0 x^m + \cdots + b_{m-1} x + b_m = b_0(x - \beta_1)\cdots(x - \beta_m).$$

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Recall: The resultant of f and g with respect to x can be defined by

$$R(f,g) = a_0^m b_0^\mu \prod_{\substack{1\le i\le m\\ 1\le j\le \mu}} \left(\beta_j - \alpha_i\right).$$

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Some properties:

$$R(f,g) = a_0^m \prod_{i=1}^{\mu} g(\alpha_i),$$
$$R(f,g) = (-1)^{\mu m} R(g,f),$$
$$R(f,pq) = R(f,p)\cdot R(f,q),$$

Another useful property:

Lemma

If we can write

$$f(x) = q(x)g(x) + r(x)$$

with polynomials q, r *and* $\nu := \deg r$*, then*

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With these properties we can prove:

Theorem

For any integer $n \geq 1$ *we have*

$$R(Q_n(x), Q_{n-1}(x)) = 2^n \binom{2n}{n}^{n-2}.$$

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Theorem

For any $n \geq 0$ we have

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For any $n \geq 0$ we have

$$R(P_n(x), Q_n(x)) = \binom{2n}{n}^{n+1}.$$

For the proof we rewrite the defining equation as

$$P_n(x)x^{n+1} = -(x+1)^{n+1}Q_n(x) + 1,$$

and use explicit formlas (in particular the leading coefficients) and the above properties.

4. Discriminants

Recall: Given a polynomial

$$\begin{aligned} f(x) &= a_m x^m + \cdots + a_1 x + a_0 \\ &= a_m (x - \theta_1) \cdots (x - \theta_m), \end{aligned}$$

$(a_m \neq 0)$, the discriminant of f is defined by

$$\begin{aligned} \mathrm{Disc}(f) &= (-1)^{\frac{m(m-1)}{2}} a_m^{-1} R(f, f') \\ &= (-1)^{\frac{m(m-1)}{2}} a_m^{m-2} \prod_{i=1}^{m} f'(\theta_i). \end{aligned}$$

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It follows that $\mathrm{Disc}(f) = 0$ iff f has multiple roots.

Theorem

For integers $n > k \geq 0$ we have

$$\mathrm{Disc}(Q_n^{(k)}(x)) = (-1)^{\varepsilon} \frac{n+k+1}{\binom{2n}{n+k}} \left(\frac{(n+k)!}{(n-k)!} \binom{2n}{n} (2n+1) \right)^{n-k-1},$$

where $\varepsilon := (n-k)(n-k-1)/2$.

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$$\mathrm{Disc}(Q_n(x)) = (-1)^{\frac{n(n-1)}{2}} (n+1)(2n+1)^{n-1} \binom{2n}{n}^{n-2}.$$

Main ingredient in proof:

$$(x+1)Q_n^{(k+1)}(x) + (n+k+1)Q_n^{(k)}(x) = (-1)^n \frac{(2n+1)!}{n!} \frac{x^{n-k}}{(n-k)!}.$$

Let x run through the zeros of $Q_n^{(k)}(x)$; use product identity for the discriminant.

Corollary

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This follows from $P_n^{(k)}(x) = (-1)^{n+k+1} Q_n^{(k)}(-x-1)$ and the discriminant identities

$$\mathrm{Disc}(f(ax+b)) = a^{m(m-1)}\mathrm{Disc}(f(x)),$$
$$\mathrm{Disc}(cf(x)) = c^{2(m-1)}\mathrm{Disc}(f(x)).$$

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Remark: Compare with the Chebyshev polynomials:

$$\mathrm{Disc}(T_n(x)) = 2^{(n-1)^2} n^n,$$
$$\mathrm{Disc}(U_n(x)) = 2^{n^2}(n+1)^{n-2}.$$

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- Are there more?
- How can we characterize them?
- How about $\mathrm{Disc}(Q_n^{(k)})$ for $k \geq 1$?

Let $n > k \geq 0$, and set $D_{k,n} := \mathrm{Disc}(Q_n^{(k)})$.

Corollary

(a) If $n \equiv k+2$ or $k+3 \pmod 4$, then $D_{k,n}$ is not the square of an integer.

Let $n > k \geq 0$, and set $D_{k,n} := \mathrm{Disc}(Q_n^{(k)})$.

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(*a*) *If* $n \equiv k + 2$ *or* $k + 3 \pmod 4$, *then* $D_{k,n}$ *is not the square of an integer.*

(*b*) *If* $n \equiv k + 1 \pmod 4$, *then for a given* k, $D_{k,n}$ *is a square for at most finitely many* n.

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(c) *If $n \equiv k \pmod 4$, then for each k there are infinitely many n such that $D_{k,n}$ is a square.*

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(c) *If* $n \equiv k \pmod 4$*, then for each* k *there are infinitely many* n *such that* $D_{k,n}$ *is a square.*

(d) *In particular,* $D_{0,n}$ *is a square if and only if* $n = 1$ *or* $n = n_j$*, where*

$$n_j := \frac{1}{8}\left((3+2\sqrt{2})^{2j+1} + (3-2\sqrt{2})^{2j+1} - 6\right), \qquad j = 1, 2, 3, \ldots$$

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Remark: Part (d) shows that $D_{0,n}$ is a square for $n = 1, 24, 840, 28560, 970224, 32959080, 1119638520, 38034750624, \ldots$

Proof is based on the formula in the previous theorem:

$$D_{k,n} = (-1)^{\varepsilon}\frac{n+k+1}{\binom{2n}{n+k}}\left(\frac{(n+k)!}{(n-k)!}\binom{2n}{n}(2n+1)\right)^{n-k-1}.$$

Proof is based on the formula in the previous theorem:

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(a) In this case, $D_{k,n} < 0$.

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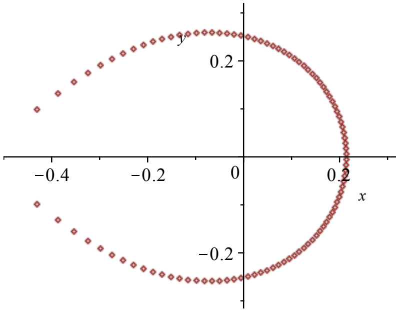
The Eneström-Kakeya theorem can then be used.

Plotting the zeros:



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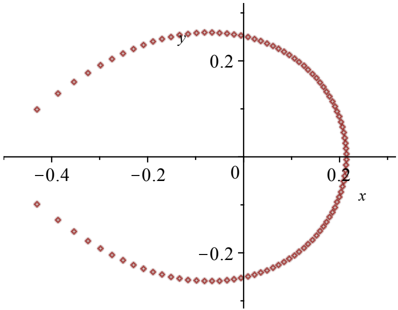
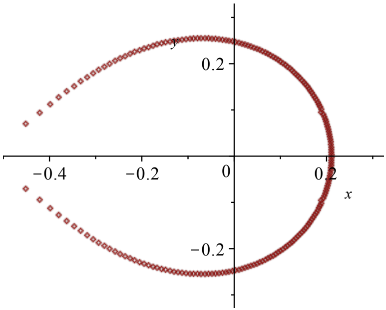


Zeros of $Q_{100}(x)$



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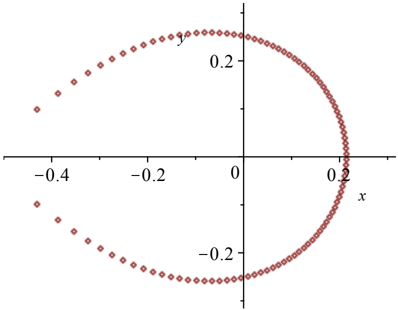

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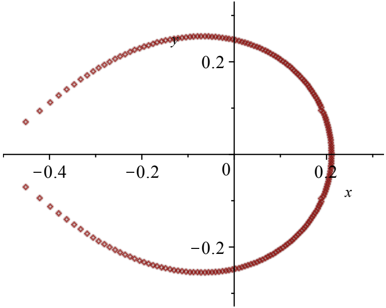

Zeros of $Q_{200}(x)$

Is there a limiting curve?

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The zeros of $Q_n(-z)$ get arbitrarily close to the "half lemniscate"

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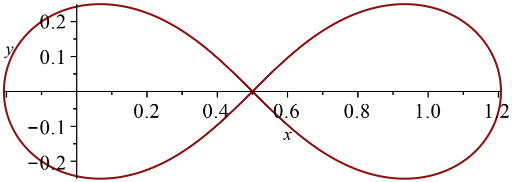

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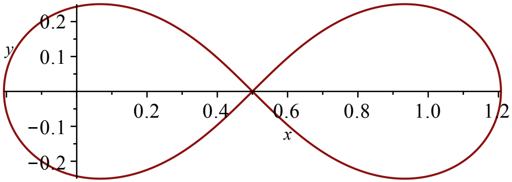
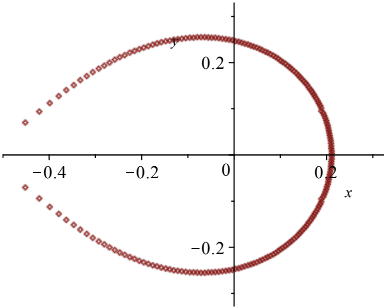


A special case of an Oval of Cassini

Compare:



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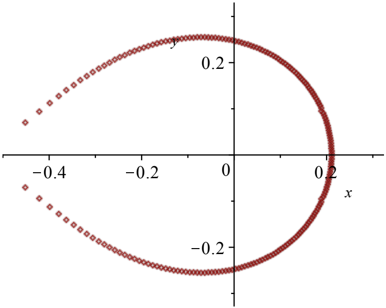

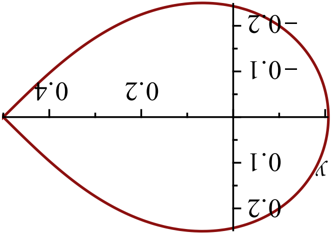

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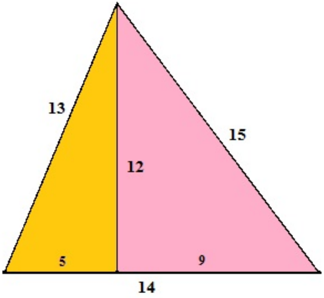
Now estimate the sum on the right.

Part II

Diophantine equations related to triangles



Joint work with

John B. Cosgrave

Dublin, Ireland

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A triangle whose side lengths and area are all integers.

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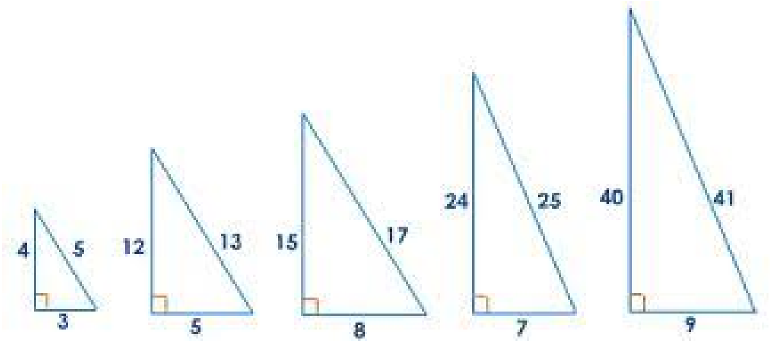

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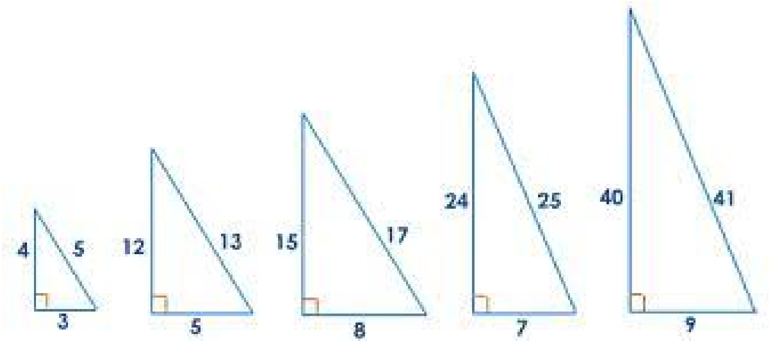

All Pythagorean triangles are Heronian.

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(17th-century German depiction)

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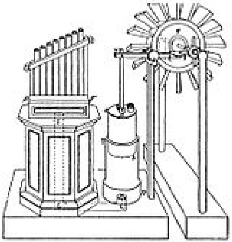

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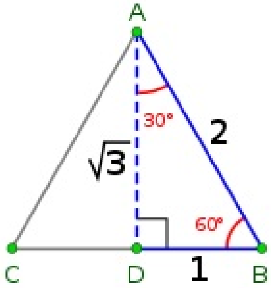

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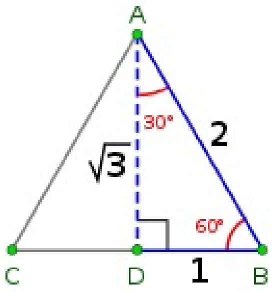

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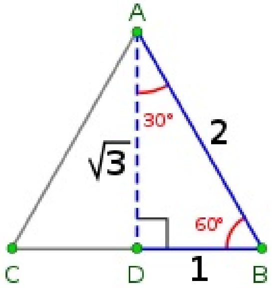

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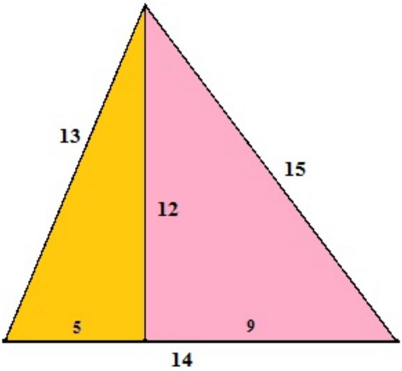
Then, how about “near equilateral"?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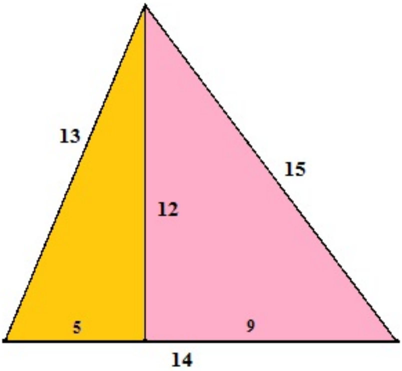
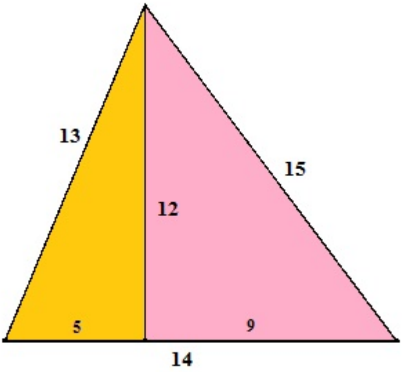


The 2nd-smallest example of a Heronian triangle of side lenghts $(n-1, n, n+1)$, after $(3, 4, 5)$.



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Are there more Heronian triangles of this type?

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This goes back to Edward Sang (1864) and R. Hoppe (1880).

Rediscovered later, for instance by L. Aubry (1911).

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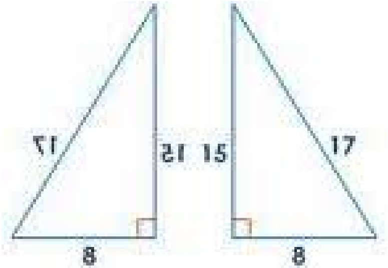

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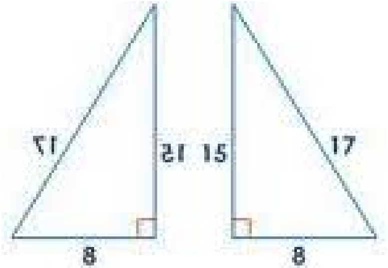

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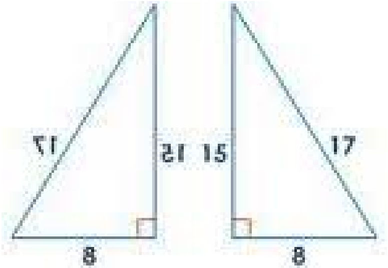

In this case we get the sequence 1, 17, 241, 3361, 46817, ..., satisfying the 3-rd order recurrence

$$p_k = 15p_{k-1} - 15p_{k-2} + p_{k-3} \qquad (k \geq 3).$$

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Thus, with Wilson's Theorem,

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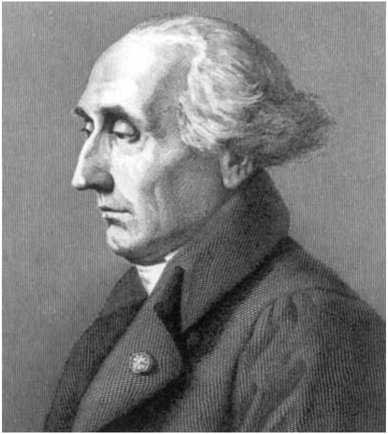
Write out the factorial $(p-1)!$, exploit symmetry mod p:

$$1\cdot 2\cdot\ldots\cdot\tfrac{p-1}{2}\tfrac{p+1}{2}\cdot\ldots\cdot(p-1) \equiv \left(\tfrac{p-1}{2}\right)!(-1)^{\frac{p-1}{2}}\left(\tfrac{p-1}{2}\right)! \pmod{p}.$$

Thus, with Wilson’s Theorem,

$$\left(\tfrac{p-1}{2}\right)!^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.$$

This was apparently first observed by Lagrange (1773).



John Wilson
1741–1793

Joseph-Louis Lagrange
1736–1813

This congruence,

$$\left(\frac{p-1}{2}\right)!^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p},$$

has the following consequences:

For $p \equiv 1 \pmod{4}$ the RHS is -1, so

$$\operatorname{ord}_p\left(\left(\frac{p-1}{2}\right)!\right) = 4 \quad \text{for} \quad p \equiv 1 \pmod{4}.$$

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What is the sign on the right?

Theorem (Mordell, 1961)

For a prime $p \equiv 3 \pmod 4$,

$$\left(\frac{p-1}{2}\right)! \equiv -1 \pmod p \quad \Leftrightarrow \quad h(-p) \equiv 1 \pmod 4,$$

where $h(-p)$ *is the class number of* $\mathbb{Q}(\sqrt{-p})$.

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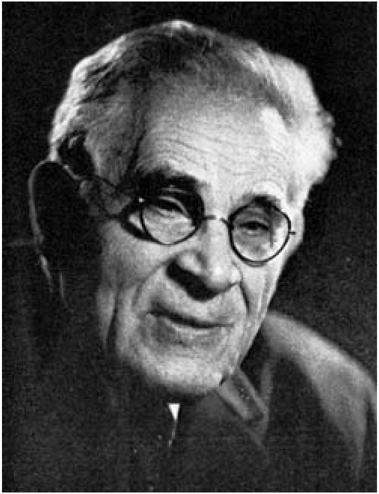
First mentioned in a book by Venkov (1937, in Russian). Discovered independently by Chowla.

This completely determines the order mod p of $\left(\frac{p-1}{2}\right)!$.



Louis J. Mordell
1888–1972

Sarvadaman Chowla
1907–1995

More generally, what can we say about the orders (mod p) of

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Here: We'll consider the case $M = 4$.

In 1828, Gauss proved the following remarkable congruence.

Let $p \equiv 1 \pmod 4$, and write $p = a^2 + b^2$ with $a \equiv 1 \pmod 4$.
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There are similar congruences for $\binom{\frac{2(p-1)}{3}}{\frac{p-1}{3}}$ (Jacobi, 1837), and others.

C. F. Gauss
1777–1855

C. G. J. Jacobi
1804–1851

Let's look at the first 30 primes $p \equiv 1 \pmod 4$:

p	$\frac{p-1}{4}!(p)$	order	p	$\frac{p-1}{4}!(p)$	order	p	$\frac{p-1}{4}!(p)$	order
5	1	1	97	20	32	197	92	98
13	6	12	101	46	100	229	168	38
17	7	16	109	7	27	233	36	116
29	23	7	113	32	28	241	130	16
37	21	18	137	90	136	257	120	32
41	13	40	149	23	148	269	258	67
53	26	52	157	145	6	277	221	276
61	19	30	173	40	86	281	157	28
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Recall: The sequence p_k is the second “Heronian" sequence.

Related to this we introduce the following terminology.

Definition

Let $p \equiv 1 \pmod 4$ be a prime. If

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- For instance, $F_2 = 17$ is indeed a Gauss prime of level 4.

All Gauss primes $p < 10^{14}$ ($p < 10^{16}$ for $\ell = 5$) and $\ell \leq 20$.

ℓ	primes
0	5 only
1–3	none
4	**17, 241, 3361, 46817, 652081,** ...
5	97, 257, 929, 262337, 200578817
6	193, 65537
7	641, 12055618177
8	3200257
9	93418448897
10	285697, 345089, 11118593
11	120833, 1249520060417
12	12289
13	1908737, 10812547073
14	114689, 8780414977
15	
16	1179649, 27590657, 2742091777
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18	786433, 3225052512257

Recall: Theorem says that $p \equiv 1 \pmod 4$ is a level-4 Gauss prime iff $p = p_k$ is a prime element of the sequence defined by $p_0 = 1$, $p_1 = 17$, $p_2 = 241$, and

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Theorem (Gauss)

For any integer $n \geq 2$ we have

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- Is this true in general?

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Recall: 46817 and 652081 are in our “Heronian" sequence.

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233456083377	$3 \cdot 11 \cdot 19 \cdot 571 \cdot 652081$	652081

Recall: 46817 and 652081 are in our “Heronian" sequence.

The next smallest solution known to us has 155 digits, with p_{133} (mentioned above) being its only prime factor $\equiv 1 \pmod 4$.

Again, for which integers $n \equiv 1 \pmod 4$ do we have

$$\left(\tfrac{n-1}{4}\right)_n! \equiv 1 \pmod n?$$

Surprisingly, there **are** other solutions with **only one** prime factor $\equiv 1 \pmod 4$. The three smallest ones are:

n	n factored	p
205479813	$3 \cdot 7 \cdot 11 \cdot 19 \cdot 46817$	46817
1849318317	$3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 46817$	46817
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All this can be fully explained.

4. Factors of p_k

All $k \leq 10^5$ for which p_k is a prime or a *probable prime*:

1	200	5 598	12 483
2	296	6 683	*13 536*
3	350	7 445	*18 006*
4	519	8 775	*18 995*
5	704	8 786	*48 773*
131	950	11 565	*93 344*

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Are there necessary conditions for p_k to be a prime?

This would speed up computations.

Using standard methods from theory of linear recurrences:

$$p_k = a_{k+1}^2 + a_k^2 \qquad (k \geq 0),$$

where the sequence $\{a_k\}$ satisfies $a_0 = 0, a_1 = 1$, and

$$a_k = 4a_{k-1} - a_{k-2} \qquad (k \geq 2).$$

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In fact, $a_k = U_k(4, 1)$, an instance of a generalized Lucas sequence. Known from the general case:

$$a_k = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^k - \left(2-\sqrt{3}\right)^k\right) \qquad (k \geq 0).$$

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Then α is a unit in $\mathbb{Q}[\sqrt{3}]$; in particular, $2-\sqrt{3} = \alpha^{-1}$.

Combining everything:

$$p_k = \frac{1}{3\alpha^m}\left(\alpha^{2m} - \alpha^m + 1\right).$$

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and the following lemma: If $n \geq 3$ and $\alpha = 2+\sqrt{3}$, then

$$\frac{1}{\alpha^{\frac{1}{2}\varphi(n)}}\Phi_n(\alpha) \in \mathbb{Z},$$

where $\varphi(n)$ is the Euler totient function.

Using this and further technical lemmas, we get

Theorem

Let $k \geq 0$ and suppose that $\gamma \geq 0$ is such that $3^\gamma \mid\mid 2k+1$. Then $p_n \mid p_k$ whenever $2n+1 \mid 2k+1$ and $3^\gamma \mid 2n+1$.

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If p_k is a prime, then $2k+1$ is a prime or a power of 3.

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An immediate consequence:

Corollary

If p_k is a prime, then $2k+1$ is a prime or a power of 3.

Example:

k	$2k+1$	k	$2k+1$
1	3	5	11
2	5	131	263
3	7	200	401
4	9	296	593

5. Back to Heronian Triangles

Recall:

$$a_k = U_k(4,1) = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^k - \left(2-\sqrt{3}\right)^k\right).$$

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It turns out: The sequence $\{n_k\}$ that gives Heronian triangles with sides $(n_k - 1, n_k, n_k + 1)$ is the companion sequence

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It follows from the theory that

$$n_k = a_{k+1} - a_{k-1} \qquad (k \geq 1).$$

Thank you

"You know, most people's favourite number is 7, but mine is 6273990103648829910048253048103855722295710049274010154829477388859173 89."