Irreducibility of Generalized Stern Polynomials

Honours Research Project with Prof. Karl Dilcher

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Outline

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2. Introduction
3. Cyclotomic Polynomials
4. Newman Polynomials, Borwein Polynomials, and Irreducibility
5. Previous Irreducibility Results for $a_{2,t}(n;z)$
6. My results

I will also prove:

Theorem

The real part of every non-trivial zero of the Riemann zeta function is $\frac{1}{2}$.

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Proof.

April Fools!! $\square$

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- Various cases proved by Schinzel and by Dilcher, Kidwai, and Tomkins
- Here we study the analogous problem for the generalized Stern polynomials.

Introduction

Definition (Stern sequence)

The Stern integer sequence, also known as Stern's diatomic series, is denoted $(a(n))_{n\geq 0}$ *and defined by* $a(0) = 0$, $a(1) = 1$, *and for* $n \geq 1$ *by*

$$a(2n) = a(n), \tag{1}$$

$$a(2n+1) = a(n) + a(n+1). \tag{2}$$

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Related to

- Stern-Brocot tree
- Calkin-Wilf tree
- Counting the rationals; random Fibonacci sequences; Fibonacci representations

Introduction

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  $0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, \ldots.$

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- The sequence $(a(n)/a(n+1))_{n\in\mathbb{N}}$ of consecutive Stern numbers gives an enumeration without repetition of the positive reduced rational numbers.
- The number $a(n+1)$ gives the number of "hyperbinary expansions" of n, i.e., the number of ways of writing n as a sum of powers of 2 without repetition.

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- and another independently by Karl Dilcher and Ken Stolarsky!!

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$$B_{2n}(z) = zB_n(z), \tag{7}$$

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$$B_{2n}(z) = zB_n(z), \tag{9}$$

$$B_{2n+1}(z) = B_n(z) + B_{n+1}(z). \tag{10}$$

We see immediately that

$$B_n(1) = a(n) \quad (n \geq 0), \tag{11}$$

and by induction that

$$B_n(2) = n \quad (n \geq 0). \tag{12}$$

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A. Schinzel proved

Theorem (Schinzel)

For all integers $n \geq 3$, $B_{2^n-3}(z)$ is irreducible.

- also proved for all primes $p < 2017$, without computation

Introduction

Further cases proved by Karl Dilcher, Mohammad Kidwai, and Hayley Tomkins, including the following theorem:

Theorem (Dilcher, Kidwai, Thomkins)

Suppose that the prime p is of the form

$$p = 2^{\nu} \pm 2^{\mu} \pm 1$$

or

$$p = 2^{\nu} \pm 2^{\mu} \pm 3$$

where $\mu \geq 1$ and $\nu \geq \mu + 9$ are integers, and the instances of "$\pm$" are independent. Then $B_p(z)$ is irreducible.

Introduction

Definition (Generalized Stern polynomials)

Let t be a fixed positive integer.
(1) The Type-1 generalized Stern polynomials $a_{1,t}(n;z)$ are polynomials in z defined by $a_{1,t}(0;z)=0$, $a_{1,t}(1;z)=1$, and for $n\geq 1$ by

$$a_{1,t}(2n;z) = za_{1,t}(n;z^t), \tag{13}$$

$$a_{1,t}(2n+1;z) = a_{1,t}(n;z^t) + a_{1,t}(n+1;z^t). \tag{14}$$

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$$a_{1,t}(2n+1; z) = a_{1,t}(n; z^t) + a_{1,t}(n+1; z^t). \tag{16}$$

(2) The Type-2 generalized Stern polynomials $a_{2,t}(n; z)$ are polynomials in z defined by $a_{2,t}(0; z) = 0$, $a_{2,t}(1; z) = 1$, and for $n \geq 1$ by

$$a_{2,t}(2n; z) = a_{2,t}(n; z^t), \tag{17}$$

$$a_{2,t}(2n+1; z) = z a_{2,t}(n; z^t) + a_{2,t}(n+1; z^t). \tag{18}$$

n	$a_{1,t}(n;z)$	$a_{2,t}(n;z)$
1	1	1
2	z	1
3	$1+z^t$	$1+z$
4	z^{t+1}	1
5	$1+z^t+z^{t^2}$	$1+z+z^t$
6	$z+z^{t^2+1}$	$1+z^t$
7	$1+z^{t^2}+z^{t^2+t}$	$1+z+z^{t+1}$
8	z^{t^2+t+1}	1
9	$1+z^{t^2}+z^{t^2+t}+z^{t^3}$	$1+z+z^t+z^{t^2}$
10	$z+z^{t^2+1}+z^{t^3+1}$	$1+z^t+z^{t^2}$
11	$1+z^t+z^{t^2}+z^{t^3}+z^{t^3+t}$	$1+z+z^{t+1}+z^{t^2}+z^{t^2+1}$
12	$z^{t+1}+z^{t^2+t+1}$	$1+z^{t^2}$
13	$1+z^t+z^{t^3}+z^{t^3+t}+z^{t^3+t^2}$	$1+z+z^t+z^{t^2+1}+z^{t^2+t}$
14	$z+z^{t^3+1}+z^{t^3+t^2+1}$	$1+z^t+z^{t^2+1}$
15	$1+z^{t^3}+z^{t^3+t^2}+z^{t^3+t^2+t}$	$1+z+z^{t^{+1}}+z^{t^2+t+1}$
16	$z^{t^3+t^2+t^1+1}$	1

Introduction

By comparing (7)-(10) with (1) and (2) we see that for $z = 1$ both sequences reduce to the Stern integer sequence $a(n)$, i.e.,

$$a_{1,t}(n;1) = a_{2,t}(n;1) = a(n) \qquad (t \geq 1, n \geq 0). \tag{19}$$

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$$a_{1,t}(n;1) = a_{2,t}(n;1) = a(n) \qquad (t \geq 1, n \geq 0). \tag{20}$$

- Table indicates that both sequences have a special structure
- For $t = 1$ the exponents in a given polynomial can coincide
- The following theorem describes the case $t \geq 2$

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Theorem

For integers $t \geq 2$ and $n \geq 0$, the coefficients of $a_{1,t}(n;z)$ and $a_{2,t}(n;z)$ are either 0 or 1. Furthermore, all exponents of z are polynomials in t with only 0 or 1 as coefficients.

Remark

This theorem and (11) *show that the number of terms of both polynomials is given by the Stern number* $a(n)$.

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Example

The hyperbinary expansions of $n = 10$ are

$$8+2, \quad 8+1+1, \quad 4+4+2, \quad 4+4+1+1, \quad 4+2+2+1+1,$$

and notice that $8+2$ is the unique binary expansion.
Observe that there are $5 = a(11) = a(10+1)$ such hyperbinary expansions.

Cyclotomic Polynomials

Definition (Root of unity)

Let K be a field and n a positive integer. An element ζ is called an nth root of unity provided $\zeta^n = 1$, that is, if ζ is a root of $z^n - 1 \in K[z]$.

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Remark

(1) If ζ_n is an nth root of unity, then $\zeta_n = e^{2\pi ik/n}$ for some $k \in \mathbb{N}$.
(2) The nth roots of unity form a cyclic subgroup of the multiplicative group K^* of nonzero elements of K.

Definition (Primitive root of unity)

An nth root of unity ζ_n is primitive if it is not a kth root of unity for any $k < n$. In other words, ζ_n is a primitive nth root of unity if it has order n in the group of nth roots of unity.

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Theorem

The primitive nth roots of unity are the elements $\{\zeta_n^k \mid \zeta_n = e^{2\pi i/n}, \gcd(k,n) = 1\}$.

Cyclotomic Polynomials

Definition (Cyclotomic polynomial)

For a positive integer n the nth cyclotomic polynomial $\Phi_n(z)$ is the unique irreducible polynomial in $\mathbb{Z}[z]$ given by

$$\Phi_n(z) = \prod_{\substack{1 \le k < n, \\ \gcd(k,n)=1}} (z - \zeta_n^k) \tag{22}$$

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Remark

(1) The roots of $\Phi_n(z)$ are precisely the primitive nth roots of unity.
(2) $\Phi_n(z)$ divides $z^n - 1$ but doesn't divide $z^k - 1$ for any positive integer $k < n$.

Cyclotomic Polynomials

We have the following identites.
If p is prime, then

$$\Phi_p(z) = 1 + z + z^2 + \ldots + z^{p-1} = \sum_{k=0}^{p-1} z^k, \tag{24}$$

and if $n = 2p$ where p is an odd prime, then

$$\Phi_{2p}(z) = 1 - z + z^2 - \ldots + z^{p-1} = \sum_{k=0}^{p-1} (-z)^k. \tag{25}$$

Cyclotomic Polynomials

Theorem (Eisenstein's Criterion)

Suppose that $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x]$*. If there exists a prime* p *for which* $p \nmid a_n$, $p \mid a_k$ *for all* $k < n$*, and* $p^2 \nmid a_0$*, then* f *is irreducible over* $\mathbb{Q}$*.*

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Lemma

$\Phi_p(z)$ *is irreducible if and only if* $\Phi_p(z+1)$ *is.*

Cyclotomic Polynomials

Theorem (Eisenstein's Criterion)

Suppose that $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x]$. If there exists a prime p for which $p \nmid a_n$, $p \mid a_k$ for all $k < n$, and $p^2 \nmid a_0$, then f is irreducible over $\mathbb{Q}$.

Lemma

$\Phi_p(z)$ is irreducible if and only if $\Phi_p(z+1)$ is.

Theorem

If p is prime, then the pth cyclotomic polynomial $\Phi_p(z)$ is irreducible.

Cyclotomic Polynomials

Proof.

Let p be prime. First notice that the binomial coefficient $\binom{p}{r}$ is divisible by p for all $0 \leq r \leq p-1$. Indeed, let

$$N = \binom{p}{r} = \frac{p!}{r!(p-r)!}.$$

Then $p! = Nr!(p-r)!$. Clearly p divides $p!$ and hence p also divides $Nr!(p-r)!$. Since p is prime, it must divide N or $r!(p-r)!$. But $r, p-r < p$ so that $p \nmid r!, (p-r)!$. Thus p divides N. Now, we have

$$\Phi_p(z+1) = \frac{(z+1)^p - 1}{z} = z^{p-1} + \binom{p}{p-2} z^{p-2} + \ldots + \binom{p}{2} z + p.$$

Every coefficient of $\Phi_p(z+1)$ except the coefficient of z^{p-1} is divisible by p by the above, and $p^2 \nmid p$. Hence by Eisenstein's Criterion $\Phi_p(z+1)$ is irreducible. Thus by the Lemma, $\Phi_p(z)$ is irreducible. □

Cyclotomic Polynomials

In fact, it is true that the nth cyclotomic polynomial is irreducible for all positive integers n.

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Exercise. □

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Exercise. [There's a nice one in *A Classical Introduction to Modern Number Theory* by Ireland and Rosen.] □

Cyclotomic Polynomials

Definition (Euler's totient function)

For a positive integer n, the number of positive integers less than n and relatively prime to n is given by Euler's totient function, $\varphi(n)$. That is, $\varphi(n) := \#\{k \in \mathbb{N} \mid k < n, \gcd(k, n) = 1\}$.

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The degree of $\Phi_n(z)$ is $\varphi(n)$.

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Proof.

By definition,

$$\Phi_n(z) = \prod_{\substack{1 \leq k < n, \\ gcd(k,n)=1}} (z - \zeta_n^k),$$

which is a product of $\varphi(n)$ factors, each having as its leading term z with coefficient 1. □

Newman Polynomials, Borwein Polynomials, and Irreducibility

Definition (Borwein polynomial, Newman polynomial)

Let

$$P = \{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \mid a_i \in \{-1, 0, 1\}\}.$$

A polynomial $f \in P$ is called a Borwein polynomial if $f(0) \neq 0$ and called a Newman polynomial if every $a_i \in \{0, 1\}$.

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- The length of a polynomial is the number of nonzero terms.
- Notice that if $a_0 = 0$, then $f(z)$ is trivially reducible. So, we will sometimes restrict to the case $a_0 = 1$.
- $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ will denote the unit circle in $\mathbb{C}$. Some but not all Newman polynomials have roots on $\mathbb{S}$, and some Newman polynomials are reducible over $\mathbb{Q}$ while others are not.

Newman Polynomials, Borwein Polynomials, and Irreducibility

Remark

In light of these new definitions, we see that for $t \geq 2$ *and* $n \geq 0$ *the polynomials* $a_{1,t}(n; z)$ *and* $a_{2,t}(n; z)$ *are Newman polynomials of length* $a(n)$.

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Theorem (Lehmer)

Given an integer $k \geq 2$*, the number of integers* n *in the interval* $2^{k-1} \leq n \leq 2^k$ *for which* $a(n) = k$ *is* $\varphi(k)$*. Furthermore, it is the same number in any subsequent interval between two consecutive powers of* 2*.*

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Given an integer $k \geq 2$, the number of integers n in the interval $2^{k-1} \leq n \leq 2^k$ for which $a(n) = k$ is $\varphi(k)$. Furthermore, it is the same number in any subsequent interval between two consecutive powers of 2.

Corollary

The number of type-1 generalized Stern polynomials of length k in the interval $[2^{k-1}, 2^k]$ is $\varphi(k)$.

Newman Polynomials, Borwein Polynomials, and Irreducibility

Theorem (Ljunggren)

If a Newman polynomial of length 3 *or* 4 *is reducible, then it has a cyclotomic factor (equivalently, it vanishes at some root of unity). That is, if*

$$f(z) = z^n + z^m + z^r + 1, \qquad n > m > r \geq 0$$

is reducible, then f *has a cyclotomic factor.*

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Conjecture (Mercer)

If a Newman polynomial of length 5 *is reducible, then it has a cyclotomic factor. That is, if*

$$f(z) = z^n + z^m + z^r + z^s + 1, \qquad n > m > r > s > 0$$

is reducible, then f *has a cyclotomic factor.*

- Mercer checked his conjecture for all Newman polynomials up to degree 24.

Corollary

The number of type-1 generalized Stern polynomials which have a cyclotomic factor in the interval $[4, 8]$ *is at most* $\varphi(3) = 2$*, in the interval* $[8, 16]$ *at most* $\varphi(4) = 2$*, and in the interval* $[16, 32]$ *at most* $\varphi(5) = 4$*.*

Theorem (Tverberg)

The trinomial

$$f(z) = z^n + z^m \pm 1 \tag{26}$$

is irreducible whenever no root of f lies on $\mathbb{S}$. If f has roots on $\mathbb{S}$, then f has a cyclotomic factor and a rational factor.

Theorem (Koley & Reddy)

Let $f(z)$ be a Newman polynomial of length 5 with a cyclotomic factor. Then f is divisible by either $\Phi_{5^\gamma}(z)$ or $\Phi_{2^\alpha 3^\beta}(z)$ for some $\alpha, \beta, \gamma \geq 1$.

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Example

We have

$$a_{1,2}(5; z) = z^4 + z^2 + 1 = (z^2 + z + 1)(z^2 - z + 1) = \Phi_3(z)\Phi_6(z).$$

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$$\begin{aligned} a_{1,2}(17;z) &= z^{16} + z^{14} + z^{12} + z^8 + 1 \\ &= (z^4 + z^3 + z^2 + z + 1)(z^4 - z^3 + z^2 - z + 1)(z^8 - z^2 + 1) \\ &= \Phi_5(z)\Phi_{10}(z)(z^8 - z^2 + 1) \end{aligned}$$

Newman Polynomials, Borwein Polynomials, and Irreducibility

Theorem (Koley & Reddy)

Suppose that f is a Borwein polynomial and $\Phi_k(z) \mid f(z)$ for some $k \in \mathbb{N}$. Then $\Phi_{k_1}(z) \mid f(z)$ for some $k_1 \mid k$ such that every prime factor of k_1 is at most $\ell(f)$, where $\ell(f)$ denotes the length of f.

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Returning to the pevious example, we see that indeed

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and $5 \mid 10$ and $5 = \ell(\Phi_5(z)), \ell(\Phi_{10}(z)) \leq \ell(a_{1,2}(17;z)) = 5$.

Theorem (Koley & Reddy)

Let $q \geq 5$ be a prime and f a primitive Newman polynomial of length q. Then $\Phi_{2q}(z) \nmid f(z)$ and $\Phi_{3q}(z) \nmid f(z)$.

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Example

We have

$$\begin{aligned} a_{1,4}(41; z) = & z^{1088} + z^{1044} + z^{1040} + z^{1024} + z^{276} \\ & + z^{272} + z^{256} + z^{64} + z^{20} + z^{16} + 1 \\ = & \Phi_{40}(z) \cdot f(z) \end{aligned}$$

for a huge polynomial $f(z)$. Indeed, $\ell(a_{1,4}(41; z)) = a(41) = 11$ is a prime greater than 5, and neither $\Phi_{22}(z)$ nor $\Phi_{33}(z)$ divides $a_{1,4}(41; z)$.

Previous Irreducibility Results for $a_{2,t}(n; z)$

- The irreducibility and factors of the type-2 generalized Stern polynomials $a_{2,t}(n; z)$ have been studied by Dilcher and Ericksen.

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- Here we state without proof their major results.

Previous Irreducibility Results for $a_{2,t}(n; z)$

- The irreducibility and factors of the type-2 generalized Stern polynomials $a_{2,t}(n; z)$ have been studied by Dilcher and Ericksen.
- Here we state without proof their major results.
- Throughout, they often employ the theorem of Lehmer mentioned earlier:

Theorem (Lehmer)

Given an integer $k \geq 2$, the number of integers n in the interval $2^{k-1} \leq n \leq 2^k$ for which $a(n) = k$ is $\varphi(k)$. Furthermore, it is the same number in any subsequent interval between two consecutive powers of 2.

Previous Irreducibility Results for $a_{2,t}(n; z)$

Since for $t \geq 2$ the $a_{2,t}(n; z)$ are all Newman polynomials, by earlier results this means we can write down all binomials, trinomials, quadrinomials, and pentanomials among the $a_{2,t}(n; z)$ for $t \geq 2$, of which there are $\varphi(2) + \cdots + \varphi(5) = 9$ different classes.

Previous Irreducibility Results for $a_{2,t}(n; z)$

Theorem

For $k \geq 1$ the binomial $a_{2,t}(3 \cdot 2^k; z)$ is irreducible if and only if $t \geq 1$ is a power of 2.

Previous Irreducibility Results for $a_{2,t}(n; z)$

Theorem

Let $k \geq 0$ and $t \geq 2$ be integers.
(a) If $t \equiv 0, 1 \mod 3$, then $a_{2,t}(5 \cdot 2^k; z)$ is irreducible.
(b) If $t \equiv 2 \mod 3$, then we have $z^2 + z + 1 \mid a_{2,t}(5 \cdot 2^k; z)$. That is, $a_{2,t}(5 \cdot 2^k; z)$ is reducible except for $a_{2,2}(5; z) = z^2 + z + 1$.

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Theorem

Let $k \geq 0$ and $t \geq 2$ be integers.
(a) If $t \equiv 0, 2 \mod 3$, then $a_{2,t}(7 \cdot 2^k; z)$ is irreducible.
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Theorem

For all integers $k \geq 0$ and $t \geq 2$, the quadrinomial $a_{2,t}(9 \cdot 2^k; z)$ is irreducible.

Previous Irreducibility Results for $a_{2,t}(n; z)$

Theorem

Let $k \geq 0$ and $t \geq 2$ be integers.
(a) If t is even, then $a_{2,t}(15 \cdot 2^k; z)$ is irreducible.
(b) If t is odd, then $a_{2,t}(15 \cdot 2^k; z)$ is divisible by $1 + z^{t^k}$.

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Theorem

Let $t \geq 2$ be an integer.
(a) If $t \equiv 2, 3 \mod 5$, then $\Phi_5(z) \mid a_{2,t}(17; z)$.
(b) If $t \equiv 1 \mod 5$, then $\Phi_5(z) \mid a_{2,t}(31; z)$.

Previous Irreducibility Results for $a_{2,t}(n; z)$

Theorem

Let $t \geq 2$ be an integer, and let $p \geq 3$ be a prime which has t as a primitive root. Then

$$(1 + z + z^2 + \cdots + z^{p-1}) \mid a_{2,t}(2^{p-1} + 1; z).$$

In particular, $a_{2,t}(2^{p-1}; z)$ is reducible in this case, with the exception of $a_{2,t}(5; z) = 1 + z + z^2$.

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Corollary

If $t \equiv 3, 5 \mod 7$, then $\Phi_7(z) \mid a_{2,t}(65; z)$.

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Theorem

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In particular, $a_{2,t}(2^p - 1; z)$ is reducible in this case.

My Results

Corollary

Due to Ljunggren, we have that every reducible type-1 generalized Stern polynomial of length 3 or 4 has a cyclotomic factor.

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If Mercer's conjecture is true, then we can say more:

Corollary

Given Mercer's conjecture, every reducible type-1 generalized Stern polynomial of length 5 has a cyclotomic factor.

My results

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$$a_{1,2}(5;z) = z^4 + z^2 + 1 = (z^2 + z + 1)(z^2 - z + 1) = \Phi_3(z)\Phi_6(z).$$

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$$a_{2,1}(7;z) = 2z^2 + z = z(2z + 1).$$

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Observation: when p is prime and $a_{1,t}(p;z)$ is not irreducible, the polynomial always has cyclotomic factors.

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$$\begin{aligned} a_{1,4}(7;z) &= z^{20} + z^{16} + 1 \\ &= (z^2 + z + 1)(z^2 - z + 1)(z^4 - z^2 + 1)(z^{12} - z^4 + 1) \\ &= \Phi_3(z)\Phi_6(z)\Phi_{12}(z)(z^{12} - z^4 + 1). \end{aligned}$$

My results

Conjecture

Let p be a prime. If $a_{1,t}(p;z)$ is not irreducible and $t = p_1^{e_1} \cdots p_r^{e_r}$ is the prime factorization of t, then

$$a_{1,t}(p;z) = \Phi_{j_1}(z) \cdots \Phi_{j_{r+2}}(z) f_1(z) \cdots f_m(z), \tag{27}$$

for at least two cyclotomic polynomials $\Phi_{j_1}, \ldots, \Phi_{j_{r+2}}$ with $\gcd(j_1, \ldots, j_{r+2}) = j_1$ and polynomials $f_1, \ldots, f_m$.

My results

Furthermore:

Conjecture

If $a_{1,t}(p;z)$ *factors completely into a product of cyclotomic polynomials*

$$a_{1,t}(p;z) = \Phi_{j_1}(z)\cdots\Phi_{j_{r+2}}(z), \quad j_1 < j_2 < \cdots < j_{r+2}, \tag{28}$$

then
(1) If $t = p_1^{e_1}$ *is a prime power and* $\gcd(j_1, t) = 1$*, then*

$$j_k = j_1 p_1^{k-1} \quad (1 \leq k-1 \leq e_1)$$

(2) If $\gcd(j_1, t) = p_i$ *for some* $1 \leq i \leq r$*, then* p_i *is not a factor of any of the* j_k*;*

My results

Conjecture (Cont'd)

(3) If $t = p_1 \cdots p_r$ is squarefree, then

$$\begin{aligned} j_2 &= p_1 j_1, \\ j_3 &= p_2 j_1, \\ &\vdots \\ j_r &= p_{r-1} j_1, \\ j_{r+1} &= p_r j_1, \\ j_{r+2} &= p_1 \cdots p_r j_1. \end{aligned}$$

(4) If $t = p_1^{e_1} \cdots p_r^{e_r}$, $r > 1$, is a product of distinct prime powers and $\gcd(t, j_1) = 1$, then ???

My results

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Corollary

The number of type-1 generalized Stern polynomials which have a cyclotomic factor is equal to the number of reducible type-1 generalized Stern polynomials.

t	n	$a_{1,t}(n;z)$	$\{j : \Phi_j \mid a_{1,t}(n;z)\}$	Case
2	5	$z^4 + z^2 + 1$	3, 6	1
	17	$z^{16} + z^{14} + z^{12} + z^8 + 1$	5, 10	1
3	3	$z^3 + 1$	2, 6	1
	73	z^{756}+too big for this margin	5, 15	1
4	7	$z^{20} + z^{16} + 1$	3, 6, 12	3
	41	$z^{1088} + z^{1044} + z^{1040} + z^{1024} + z^{276} + z^{272} + z^{256} + z^{64} + z^{20} + z^{16} + 1$	40 (up to 10,000)	3
5	3	$z^5 + 1$	2, 10	1
	5	$z^{25} + z^5 + 1$	5, 15	1
6	3	$z^6 + 1$	4, 12	2
	31	$z^{1554} + z^{1548} + z^{1512} + z^{1296} + 1$	5, 10, 15, 30	3
7	3	$z^7 + 1$	2, 14	1
	7	$z^{56} + z^{49} + 1$	3, 21	1
	17	$z^{2401} + z^{399} + z^{392} + z^{343} + 1$	5, 35	1
8	5	$z^{64} + z^8 + 1$	3, 6, 12, 24	1
9	3	$z^9 + 1$	2, 6, 18	1

t	n	$a_{1,t}(n;z)$	$\{j : \Phi_j \mid a_{1,t}(n;z)\}$	Case
10	3	$z^{10}+1$	4, 20	2
	7	$z^{110}+z^{100}+1$	3, 6, 15, 30	3
11	3	$z^{11}+1$	2, 22	1
	5	$z^{121}+z^{11}+1$	3, 33	1
12	3	$z^{12}+1$	8, 24	2
13	3	$z^{13}+1$	2, 26	1
	7	$z^{182}+z^{169}+1$	3, 39	1
14	3	$z^{14}+1$	4, 28	2
	5	$z^{196}+z^{14}+1$	3, 6, 21, 42	3
15	3	$z^{15}+1$	2, 6, 10, 30	3?
16	7	$z^{272}+z^{256}+1$	3, 6, 12, 24, 48	1
17	3	$z^{17}+1$	2, 34	1
	5	$z^{289}+z^{17}+1$	3, 51	1
18	3	$z^{18}+1$	4, 12, 36	2

Table: Classification of $a_{1,t}(n;z)$ by cyclotomic factors

My Results

- Notice that the first instance of Case 4 doesn't occur until $t = 2^2 3^2 = 36$.

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- Since the "size" of these polynomials grows very quickly, it becomes computationally expensive to factor them for large n and t.
- Use cluster for this

n	$a_{1,t}(n;z)$
17	$z^{t^4} + z^{t^3+t^2+t} + z^{t^3+t^2} + z^{t^3} + 1$
18	$z^{t^4+1} + z^{t^3+t^2+1} + z^{t^3+1} + z$
19	$z^{t^4+1} + z^{t^4} + z^{t^3+t^2} + z^{t^3+1} + z^{t^3} + z^{t} + 1$
20	$z^{t^4+t+1} + z^{t^3+t+1} + z^{t+1}$
21	$z^{t^4+t^2} + z^{t^4+t} + z^{t^4} + z^{t^3+t} + z^{t^3} + z^{t^2} + z^{t} + 1$
22	$z^{t^4+t^2+1} + z^{t^4+1} + z^{t^3+1} + z^{t^2+1} + z$
23	$z^{t^4+t^2+t} + z^{t^4+t^2} + z^{t^4} + z^{t^3} + z^{t^2+t} + z^{t^2} + 1$
24	$z^{t^4+t^2+t+1} + z^{t^2+t+1}$
25	$z^{t^4+t^3} + z^{t^4+t^2+t} + z^{t^4+t^2} + z^{t^4} + z^{t^2+t} + z^{t^2} + 1$
26	$z^{t^4+t^3+1} + z^{t^4+t^2+1} + z^{t^4+1} + z^{t^2+1} + z$
27	$z^{t^4+t^3+t} + z^{t^4+t^3} + z^{t^4+t^2} + z^{t^4+t} + z^{t^4} + z^{t^2} + z^{t} + 1$
28	$z^{t^4+t^3+t+1} + z^{t^4+t^3+t} + z^{t^4+t^3} + z^{t^4+t} + z^{t^4} + z^{t} + 1$
29	$z^{t^4+t^3+t^2} + z^{t^4+t^3+t} + z^{t^4+t^3} + z^{t^4+t} + z^{t^4} + z^{t} + 1$
30	$z^{t^4+t^3+t^2+1} + z^{t^4+t^3+1} + z^{t^4+1} + z$
31	$z^{t^4+t^3+t^2+t} + z^{t^4+t^3+t^2} + z^{t^4+t^3} + z^{t^4} + 1$
32	$z^{t^4+t^3+t^2+t+1}$

My results

Proposition

For $t > 0$ *and* $m \geq 1$,

$$a_{1,t}(2^m; z) = z^{t^{m-1}+t^{m-2}+\cdots+t+1} \tag{29}$$

is trivially reducible.

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Proof.

First note that $a(n) = 1$ if and only if $n = 2^m$, $m \geq 0$. Furthermore, $a_{1,t}(2^m; z)$ is a positive power of z for every $m \geq 1$. □

References

- K. Dilcher, M. Kidwai and H. Thomkins, *Zeros and irreducibility of Stern polynomials*, Publ. Math. Debrecen. 90 (2017), no. 3-4, 407-433.
- K. Dilcher and L. Ericksen, *Generalized Stern polynomials and hyperbinary representations*, Bull. Pol. Acad. Sci. Math. 65 (2017), no.1, 11-28
- K. Dilcher and L. Ericksen, *Factors and irreducibility of generalized Stern polynomials*, Integers 15 (2015), Paper no. A50, 17 pp.
- I. Mercer, *Newman polynomials, Reducibility, and Roots on the Unit Circle*, Integers 12 (2012), 503-519.
- B. Koley and A. S. Reddy, *Cyclotomic factors of Borwein polynomials*, Bull. Aus. Math. Soc. (2019) (To appear)
- H. Tverberg, *On the irreducibility of the trinomials*, Math. Scand. 8 (1960), 121-126.

Thanks!

- Thanks!
- Questions, comments, suggestions?