A Probabilistic Approach to some Elliptic Functions

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1: D. Romik's paper about Jacobi θ_3 Taylor coefficients

2: Dedekind eta function

3: Eta Products

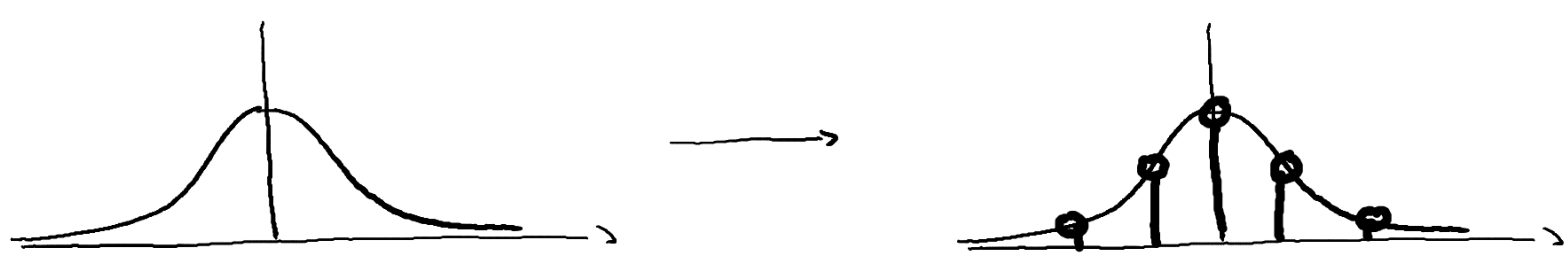
The discrete normal distribution

- $$\Pr\{X(\omega)=n\} = \frac{1}{\theta_3(q)} \cdot q^{n^2}, \quad 0<q<1$$

with $$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{Jacobi } \theta_3$$

- a sampled version of the continuous normal

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad x\in\mathbb{R}$$



The discrete normal distibution

- $k \in (0,1)$ elliptic modulus

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

elliptic integral

- Jacobi's identity

$$\frac{1}{\sqrt{\frac{2}{\pi}K(k)}}\,\theta_3\left(e^{-\pi\frac{K'(k)}{K(k)}}\right) = 1$$

$$\begin{cases} K'(k) = K(k') \\ k' = \sqrt{1-k^2} \end{cases}$$

- a natural parameterization :

$$\Pr\{X = n\} = \frac{1}{\sqrt{\frac{2}{\pi}K(k)}}\ \underbrace{e^{-\pi n^2\frac{K'(k)}{K(k)}}}_{q^{n^2}} \quad ; \quad k \in (0,1)$$

$$q^{n^2}\ ,\quad q = e^{-\pi\frac{K'(k)}{K(k)}}$$

With $$\varphi(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$$

Romik observed

$$\varphi'(1) = -\frac{1}{4}\varphi(1)$$

$$\varphi''(1) = \frac{1}{16}\varphi(1)(3+\Omega)$$

$$\varphi'''(1) = -\frac{1}{64}\varphi(1)(15+15\Omega)$$

$$\varphi^{(4)}(1) = \frac{1}{256}\varphi(1)(105+210\Omega-\Omega^2)$$

$$\Omega = \frac{\Gamma\left(\frac{1}{4}\right)^8}{32\pi^4}, \qquad \varphi(1) = \frac{\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\,\pi^{3/4}}$$

D. Romik's identity I

$$\underbrace{\frac{1}{\theta_3(e^{-\pi})}\sum_{p\in\mathbb{Z}} p^{2n} e^{-\pi p^2}}_{\mathbb{E}\, X_{\theta_3}^{2n}} = \frac{1}{(4\pi)^n}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} \frac{(2n)!}{2^{n-2j}(4j)!(n-2j)!} \times d(j)\, \Omega^{j}$$

$$\Omega = \frac{\Gamma^8(\frac{1}{4})}{32\pi^4}$$

with $d(n) = \underset{n=0}{1}, -1, 51, 849, -26199, 1341999, \dots$

an integer sequence

D. Romik's identity II

$$\frac{1}{\theta_3(e^{-\pi})} \sum_{p \in \mathbb{Z}} e^{-\pi p^2} H_{2n}(\sqrt{2\pi}\cdot p) = \begin{cases} 2^{2n} \cdot \Phi^{\frac{n}{2}} d\left(\frac{n}{2}\right) & , n \text{ even} \\ 0 & , n \text{ odd} \end{cases}$$

$$\Phi = \frac{\Gamma^8\left(\frac{1}{4}\right)}{2^7 \pi^4}$$

$$\begin{cases} H_n \text{. Hermite polynomials} \\ \sum H_n(x) \frac{w^n}{n!} = e^{2xw - w^2} \end{cases}$$

Romik's Identities I and II

$$\underbrace{\frac{1}{\theta_3(e^{-\pi})} \sum_{p\in\mathbb{Z}} p^{2n} e^{-\pi p^2}}_{\mathbb{E} X_{\theta_3}^{2n}} = \frac{1}{(4\pi)^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n)!}{2^{n-2j}(4j)!(n-2j)!} \times d(j)\, \Omega^j$$

$$\underbrace{\frac{1}{\theta_3(e^{-\pi})} \sum_{p\in\mathbb{Z}} e^{-\pi p^2} H_{2n}(\sqrt{2\pi}\cdot p)}_{\mathbb{E}\, H_{2n}(\sqrt{2\pi}\, X_{\theta_3})} = \begin{cases} 2^{2n} \cdot \Phi^{\frac{n}{2}}\, d\left(\frac{n}{2}\right), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Where does the sequence $d(n)$ come from ?

$$U(t) = \frac{{}_2F_1\left(\begin{matrix} 3/4, 3/4 \\ 3/2 \end{matrix} ; 4t\right)}{{}_2F_1\left(\begin{matrix} 1/4, 1/4 \\ 1/2 \end{matrix} ; 4t\right)}$$

$$V(t) = \sqrt{{}_2F_1\left(\begin{matrix} 1/4, 1/4 \\ 1/2 \end{matrix} ; 4t\right)}$$

$$\sum_{n \geq 0} \frac{d(n)}{2^n (2n)!} t^n U(t)^{2n} = V(t)$$

is a generating function for $\{d(n)\}$

Where does the sequence $d(n)$ come from ?

$$U(t) = \frac{{}_2F_1\left(\begin{matrix}3/4,\,3/4\\3/2\end{matrix};4t\right)}{{}_2F_1\left(\begin{matrix}1/4,\,1/4\\1/2\end{matrix};4t\right)}$$

$$V(t) = \sqrt{{}_2F_1\left(\begin{matrix}1/4,\,1/4\\1/2\end{matrix};4t\right)} = \sum_{n\geq 0} \frac{v(n)}{2^n\,(2n)!}\,t^n$$

$$d(n) = v(n) - \sum_{k=1}^{n-1} r(n,k)\,d(k) \qquad (n \geq 1)$$

with $r(n,k) = 2^{n-k}\,\dfrac{2n!}{2k!}\,[t^{n-k}]\,U(t)^{2k}$

$$\begin{array}{cc} & k \longrightarrow \\ n \downarrow & \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 48 & 1 & 0 & \cdots \\ 7584 & 240 & 1 & \cdots \\ & & & \ddots \end{bmatrix} \end{array}$$

is a recurrence relation for $\{d(n)\}$

What are the cumulants of X_{θ_3} ?

$$(a;q)_\infty = \prod_{n\geq 0} (1-aq^n)$$

- $$\theta_3(z;q) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{i2nz}$$

$$\mathbb{E}\, e^{zX_{\theta_3}} = \frac{1}{\theta_3(0;q)} \sum_{n\in\mathbb{Z}} q^{n^2} e^{nz} = \frac{\theta_3(\frac{z}{2i};q)}{\theta_3(0;q)}$$

- Jacobi's triple product identity

$$\theta_3(z;q) = (q^2,q^2)_\infty (-qe^{i2z}, q^2)_\infty (-qe^{-i2z}, q^2)_\infty$$

- $$\frac{\theta_3(\frac{z}{2i};q)}{\theta_3(0;q)} = \prod_{p\geq 0} \frac{(1+e^z q^{2p+1})(1+e^{-z}q^{2p+1})}{(1+q^{2p+1})(1+q^{2p+1})}$$

What are the cumulants of X_{θ_3} ?

- $$\operatorname{Log} \mathbb{E} e^{z X_{\theta_3}} = \sum_{n \geq 1} \frac{z^{2n}}{2n!} \kappa_{2n}$$

Lambert series

with
$$\begin{cases} \kappa_{2n} = 2 \sum_{k \geq 1} \frac{(-1)^{k-1} k^{2n-1}}{q^{-k} - q^{k}}, & n \geq 1 \\ \kappa_{2n+1} = 0, & n \geq 1 \end{cases}$$

- $q = e^{-\pi c} \Rightarrow \kappa_{2n} = \sum_{k \geq 1} \frac{(-1)^{k-1} k^{2n-1}}{\sinh(k\pi c)}$

A generating function for the cumulants

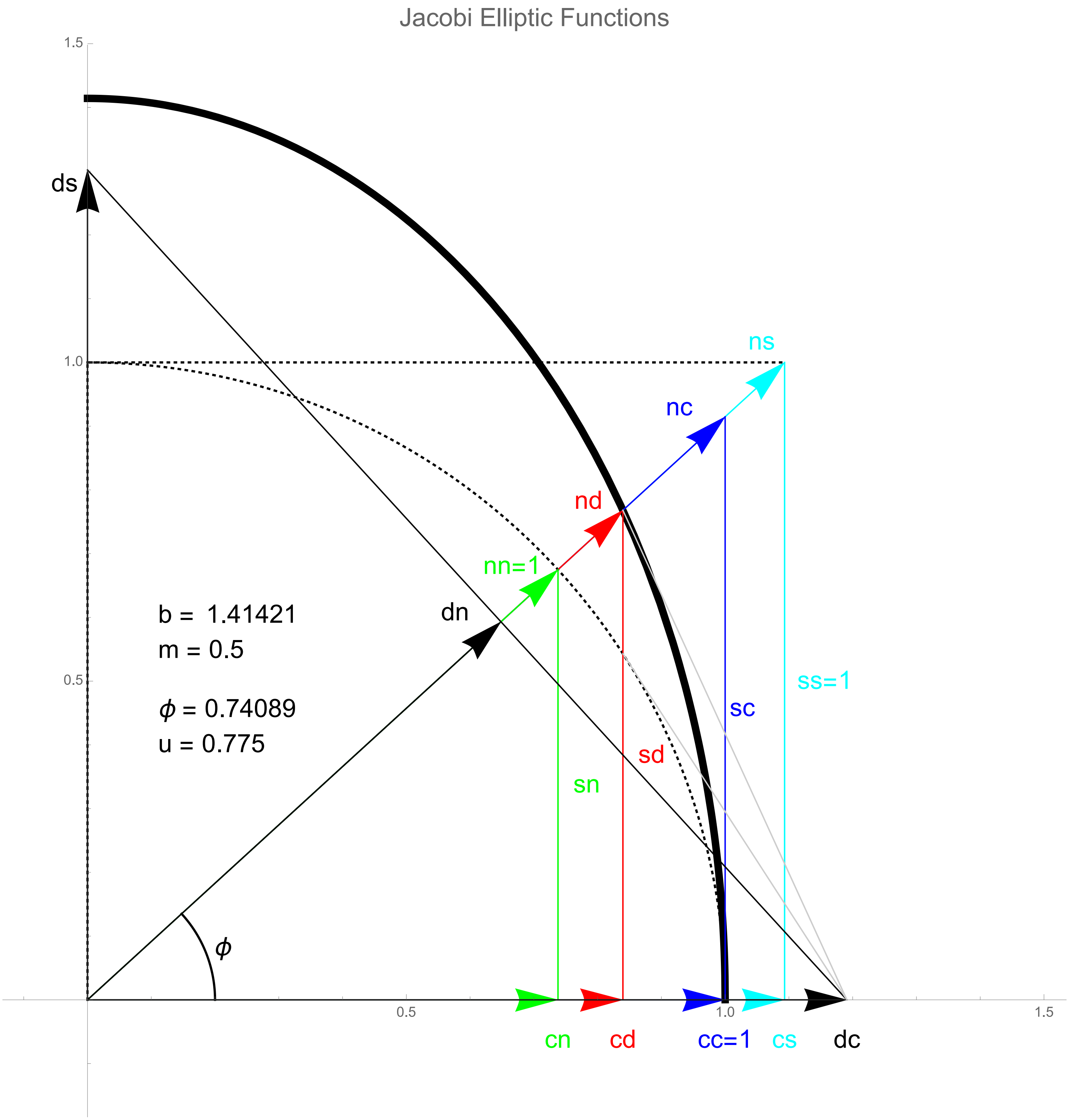
$$\mathrm{sd}^2(u,k) = -\frac{1}{(kK')^2} \sum_{m\geq 1} \frac{(-1)^m\, 2^{2m}}{\theta_3^{4m+4}(q)}\, \kappa_{2m+2}\, \frac{u^{2m}}{2m!}$$

Jacobi Elliptic function

A continued fraction representation $\left(k=\frac{1}{\sqrt{2}} \text{ only}\right)$

$$\sum_{n\geq 1} \kappa_{4n}\, x^n = \cfrac{\beta_0 x}{1+\cfrac{\beta_1 x}{1+\cfrac{\beta_2 x}{1+\cdots}}}$$

with
$$\begin{cases} \beta_n = 2\kappa_4\, n(n+1)(2n+1)^2 & n \geq 1 \\ \beta_0 = \kappa_4 \end{cases}$$


Jacobi Elliptic Functions
1.5
1.0
0.5
0.5
1.0
1.5
ds
ns
nc
nd
nn=1
dn
b = 1.41421
m = 0.5
ϕ = 0.74089
u = 0.775
ss=1
sc
sd
sn
ϕ
cn
cd
cc=1
cs
dc

The cumulants as Eisenstein series

- with $c = \dfrac{K'(k)}{K(k)}$, $q = e^{-\pi c}$

$$\kappa_{2n} = \frac{(-1)^{n+1}(2n-1)!}{\pi^{2n}} \sum_{n_1, n_2 \in \mathbb{Z}} \frac{1}{\left((2n_1-1) + ic(2n_2-1)\right)^{2n}} .$$

Schett polynomials $\quad X_n(x,y,z)$

$$\begin{cases} X_n = \left(yz\dfrac{d}{dx} + zx\dfrac{d}{dy} + xy\dfrac{d}{dz} \right) X_{n-1} \\ X_0(x,y,z) = x \end{cases}$$

$x=0,\ y=k,\ z=\imath k'$

- $X_0 = x$,
- $X_1 = yz$,
- $X_2 = x(y^2+z^2)$
- $X_3 = yz(4x^2+y^2+z^2)$

- $X_1 = \imath k k'$
- $X_3 = \imath k k' (2k^2-1)$
- $X_5 = \imath k k' (k^4 - 14(kk')^2 + k'^4)$

Define $\quad P_{2p}(k) = \sum_{n=0}^{p-1} \binom{2p}{2n+1} X_{2n+1}(0,k,\imath k') X_{2p-2n-1}(0,k,\imath k')$

Schett Polynomials

- $P_0(k) = 0$,
- $P_2(k) = -2(kk')^2$
- $P_4(k) = -8(kk')^2(2k^2-1)$
- $P_6(k) = -16(kk')^2(2-17k^2+17k^4)$

<u>Theorem</u> : The cumulants of X_{θ_3} are

$$\begin{cases} \kappa_{2n} = (-1)^{n-1}! \cdot \left(\dfrac{\theta_3^2(q)}{2}\right)^{2n} \cdot P_{2n-2}(k) , & n \geq 2 \\ \kappa_2 = \left(\dfrac{\theta_3^2(q)}{2}\right)^2 \cdot \left[\dfrac{E(k)}{K(k)} - (k')^2\right] = \sigma^2 \end{cases}$$

Bell Polynomials

$$\sum \frac{\mu_n}{n!} z^n = \exp\left(\sum \frac{\kappa_n}{n!} z^n\right)$$

The moments μ_n can be deduced from the cumulants

- $\mu_1 = \kappa_1$
- $\mu_2 = \kappa_2 + \kappa_1^2$
- $\mu_3 = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3$
- $\mu_4 = \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4$

- $\mu_1 = 0$
- $\mu_2 = \kappa_2$
- $\mu_3 = 0$
- $\mu_4 = \kappa_4 + 3\kappa_2^2$

- $\mu_n = \sum_{k=1}^{n} B_{n,k}(\kappa_1, \ldots \kappa_{n-k+1})$ homogenous

$\uparrow$ incomplete Bell polynomials

Insert the cumulants :

$$\begin{cases} \kappa_{2n} = (-1)^{n-1} \cdot \left(\frac{\theta_3^2(q)}{2}\right)^{2n} \cdot P_{2n-2}(k) \,, & n \geqslant 2 \\ \kappa_2 = \left(\frac{\theta_3^2(q)}{2}\right)^2 \cdot \left[\frac{E(k)}{K(k)} - (k')^2\right] \end{cases}$$

in the Bell polynomials ?

. if $\kappa_2 = 0$: $\quad \mu_{2n} = \left(\frac{\theta_3^2(q)}{2}\right)^{2n} \cdot \underbrace{R_{2n}(k)}_{\text{"Bell tansform" of } P_{2n}}$

$$\Downarrow$$

Integer coefficients

. but $\kappa_2 \neq 0$!

Consider

$$Z = X_{\theta_3} + \iota N_{\sigma^2}$$

$\downarrow$ Gaussian, variance $= \sigma^2$

- $\sigma_Z^2 = \sigma_{X_{\theta_3}}^2 - \sigma^2 = 0$ for $\sigma^2 = \sigma_{X_{\theta_3}}^2$

- $n \geq 2$: $\kappa_{2n}(Z) = \kappa_{2n}(X_{\theta_3}) + \iota^{2n} \underbrace{\kappa_{2n}(N_{\sigma^2})}_{=0}$

$$\Rightarrow \kappa_{2n}(Z) = \kappa_{2n}(X_{\theta_3})$$

- compute the moments of Z instead of X_{θ_3}

- works for any value of q !

Theorem : With $q = e^{-\pi \frac{K'(k)}{K(k)}}$, $\sigma^2 = \left[\frac{\theta_3^2(1)}{2}\right]^2 \left[\frac{E(k)}{K(k)} - k'^2\right]$

and $R_{2n}(k)$ integer coefficients

$$\frac{1}{\theta_3(q)} \sum_{p\in\mathbb{Z}} p^{2n} q^{p^2} = \sum_{j=0}^{n} \binom{2n}{2j} \left(\frac{\theta_3^2(q)}{2}\right)^{2j} R_{2j}(k) \left(\frac{\iota\sigma}{2}\right)^{2n-2j} \frac{2n-2j!}{n-j!}$$

or equivalently

$$\frac{1}{\theta_3(q)} \sum_{p\in\mathbb{Z}} q^{p^2} H_{2n}\left(\frac{p}{\sigma\sqrt{2}}\right) = \left(\frac{\theta_3^2(q)}{2\sigma}\right)^{2n} R_{2n}(k)$$

. the special case $k = \frac{1}{\sqrt{2}}$ is Romik's identity

Back to the standard ($\kappa = \frac{1}{\sqrt{2}}$) case :

the cumulants κ_{2n} are related to the sequence

A144849 as

$$\kappa_{4n} = 2 . (-12)^{n-1} A_{n-1}$$

$A_0 = 1, \quad A_1 = 6, \quad A_2 = 336, \quad A_4 = 77616 \; ---$

" number of increasing bilabeled strict binary trees with $4n+2$ labels "

that satisfies $\quad A_{n+1} = \sum_{j=0}^{n} \binom{4n+4}{4j+2} A_j . A_{n-j}$

$$\Rightarrow \quad \kappa_{4n} = -6 \sum_{j=0}^{n-2} \binom{4n-4}{4j+2} \kappa_{4j+4} . \kappa_{4n-4j-4}$$

1, 6, 336, 77616, 50916096, 76307083776, 226653840838656, 1207012936807028736, 106962776783084867 42016, 1489000904570445412097064 96, 3110043187741674836967136690176, 9388520612426930179033801580190105 6, 397085954981441691251999257190301538 7136 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Denoted by \beta_n in Lomont and Brillhart (2011) on page xiii.
Gives the number of Increasing bilabeled strict binary trees with 4n+2 labels. - Markus Kuba, Nov 18 2014

REFERENCES J. S. Lomont and J. Brillhart, Elliptic Polynomials, Chapman and Hall, 2001; see p. 86.

LINKS N. J. A. Sloane, Table of n, a(n) for n = 0..100
O. Bodini, M. Dien, X. Fontaine, A. Genitrini, and H. K. Hwang, Increasing Diamonds, in LATIN 2016: 12th Latin American Symposium, Ensenada, Mexico, April 11-15, 2016, Proceedings Pages pp 207-219 2016 DOI 10.1007/978-3-662-49529-2_16; Lecture Notes in Computer Science Series Volume 9644.
Markus Kuba, Alois Panholzer, Combinatorial families of multilabelled increasing trees and hook-length formulas, arXiv:1411.4587 [math.CO], (17-November-2014).
Tanay Wakhare, Christophe Vignat, Taylor coefficients of the Jacobi theta3(q) function, arXiv:1909.01508 [math.NT], 2019.
Eric Weisstein's World of Mathematics, Lemniscate Constant

FORMULA E.g.f.: sl(x)^2 = 2 Sum_{k>=0} (-12)^k * a(k) * x^(4*k + 2) / (4*k + 2)! where sl(x) = sin lemn(x) is the sine lemniscate function of Gauss. - Michael Somos, Apr 25 2011
a(0) = 1, a(n + 1) = Sum_{j=0..n} binomial(4*n + 4, 4*j + 2) * a(j) * a(n - j).
G.f.: 1 / (1 - b(1)*x / (1 - b(2)*x / (1 - b(3)*x / ...))) where b(n) =

A recurrence for the moments $\left(k = \frac{1}{\sqrt{2}} \text{ only}\right)$

$$\begin{cases} \kappa_{4n} = -6 \sum_{j=0}^{n-2} \binom{4n-4}{4j+2} \kappa_{4j+4} \cdot \kappa_{4n-4j-4} \\ \mu_{2n} = \kappa_{2n} + \sum_{m=1}^{2n-1} \binom{2n-1}{m-1} \kappa_m \cdot \kappa_{2n-m} \end{cases}$$

or use $\mu_n = \sum_{\pi \vdash n} \kappa_\pi$ with $\kappa_\pi = \prod \kappa_{\pi_j}$

for $\pi = (\pi_1 \ldots \pi_k)$

A recurrence for the moments ; any k

$$\begin{cases} \kappa_{2n+2} = (k'^2 - k^2)\,\theta_3^4(q)\,\kappa_{2n} - 6\displaystyle\sum_{\nu=1}^{n-2} \binom{2n-2}{2\nu} \kappa_{2\nu+2}\,\kappa_{2n-2\nu} \\ \mu_n = \kappa_n + \displaystyle\sum_{m=1}^{n-1} \binom{n-1}{m-1} \kappa_m\,\mu_{n-m} \end{cases}$$

using S. Wrigge, Calculation of the Taylor series expansion of the Jacobian elliptic function $\mathrm{sn}(x,k)$, Mathematics of Computation, 36, 154, 1981

Other integer sequences

- Romik's sequence

$$d(n) = 1, -1, 51, 849, -26199, \dots$$

is $R_{4n}\left(\frac{1}{\sqrt{2}}\right)$

- we look at $\quad d_\ell(n) = R_{4n}\left(\frac{1}{\sqrt{\ell}}\right) \qquad 3 \le \ell \le 7$

$$\Rightarrow \boxed{k = \frac{1}{\sqrt{\ell}}}$$

ℓ	α_m	$\alpha_m d_\ell(m)$
$1/\sqrt{3}$	$\left(\frac{3}{2}\right)^{2m}$	1, 3, 7, 2953, 291969, 12470011, ...
$1/2$	$\frac{2^{3m}}{2}$	1, 29, 43, 116171, 78138169, ...
$1/\sqrt{5}$	$\frac{5^{2m}}{2^{3m}}$	1, 17, 105, 4521, 1802457, ...
$1/\sqrt{6}$	$\frac{2^m 3^{3m}}{5}$	1, 123, 8059, 724877, 1686624921, ...
$1/\sqrt{7}$	$\frac{14^{2m}}{3}$	1, 97, 5959, 293923, 294067681, ...

Elliptic Functions as continuous probability distributions

Dedekind's eta $\quad \eta(x) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) \qquad q = e^{i 2\pi x}$

is a weight $1/2$ modular form:

- $\eta(x+1) = e^{\frac{i\pi}{12}} \eta(x)$
- $\eta\left(\frac{i}{x}\right) = \sqrt{x}\; \eta(ix)$

Euler's identity: $\quad \eta(ix) = \sum_{n \in \mathbb{Z}} (-1)^n e^{-\frac{\pi}{12}(6n+1)^2}$

Larry Glasser computed recently

- $$\int_0^\infty e^{-xy}\,\eta(ix)\,dx = \sqrt{\frac{\pi}{y}}\;\frac{\sinh 2\sqrt{\frac{\pi y}{3}}}{\cosh\sqrt{3\pi y}}, \qquad y \geq 0$$

- $$\int_0^\infty e^{-xy}\,\eta^3(ix)\,dx = \operatorname{sech}\sqrt{\pi y}, \qquad y \geq 0$$

- suggests the definition:

$$\eta_k(x) = \eta^k(ix)$$

Theorem: For $k \geq 1$ integer,

$$f_{\eta_k}(x) = \frac{1}{A_k} \eta^k(ix), \quad x > 0$$

is a probability density, with normalization constant

$$A_k = \int_0^\infty \eta^k(ix)\, dx$$

The only values we know:

$$A_1^{-1} = \frac{\sqrt{3}}{2\pi}, \quad A_2^{-1} = \mathrm{Log}\left(1 + \sqrt{3} + \sqrt{3 + 2\sqrt{3}}\right)$$

$$A_3^{-1} = 1, \quad A_4^{-1} = \frac{3\,\Gamma^3\left(\frac{2}{3}\right)}{2^{2/3}\,\pi}$$

$$A_6^{-1} = 8\pi \left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right)^2$$

<u>Theorem</u>: for $X_{\eta_k} \sim f_{\eta_k}$, all the moments

$$\mathbb{E} X_{\eta_k}^p = m_k(p) \quad \text{exist} \quad (p \in \mathbb{Z}\,!)$$

and satisfy

- $m_k(p) = m_k\left(\frac{k}{2} - p - 2\right)$
- for $X_4 \sim f_{\eta_4}$,

$$X_4 \sim \frac{1}{X_4}$$

as a consequence of modularity:

$$\eta_k(z) = z^{-k/2}\, \eta_k\left(\frac{1}{z}\right)$$

For example:

$$\int_0^\infty \eta^6(ix)\,dx = \int_0^\infty x\eta^6(ix)\,dx = \frac{1}{8\pi}\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)^2$$

$$\Rightarrow \mathbb{E} X_{\eta_6} = 1.$$

Elliptic Functions as Cumulative Distribution Functions

Consider
$$F_k(z) = \prod_{m\geq 1}\left(1 - e^{-2\pi m z}\right)^k = e^{\frac{\pi k z}{12}}\,\eta^k(iz) \qquad z \geq 0$$

This is a c.d.f. :

$$F_k(z) = \Pr\{X \leq z\}$$

for some random variable X.

A general result

<u>Theorem</u> Consider

- $\{a_i\}$ sequence of positive numbers
- $\{E_i\}$ sequence of iid exponential random variables:

$$\Pr\{E_i \leq z\} = 1 - e^{-z}$$

The random variable

$$X = \max_{i \geq 1} \left\{ \frac{E_i}{a_i} \right\}$$

has cumulative distribution function

$$\Pr\{X \leq z\} = \prod_{i \geq 1} \left(1 - e^{-a_i z}\right)$$

For example:

- $a_k = 2\pi . k \quad , k \geq 1$

$$X = \max_{k \geq 1} \left\{ \frac{E_k}{2\pi k} \right\}$$

has c.d.f. $\quad Pr\{X \leq z\} = \prod_{m \geq 1} (1 - e^{-2\pi m z})$

- $\begin{cases} a_1 = \dots = a_k = 2\pi \\ a_{k+1} = \dots = a_{2k} = 2(2\pi) \\ \vdots \\ a_{(p-1)k+1} = \dots = a_{pk} = p(2\pi) \end{cases} \Rightarrow \quad X = \max \left\{ \frac{E_k}{a_k} \right\}$

$$Pr\{X \leq z\} = \prod_{m \geq 1} (1 - e^{-2\pi m z})^k$$

Example: the hyperbolic secant distribution

$$f_X(x) = \operatorname{sech}(\pi x)$$

<u>Theorem</u> : if $X \sim \operatorname{sech}(\pi x)$ then

$$X = \sqrt{A_{\eta_3}} \cdot N$$

↳ Gaussian, variance $= \frac{1}{2\pi}$

<u>proof</u> : $\mathbb{E} e^{izX} = \operatorname{sech}\left(\frac{z}{2}\right) = \int e^{-x\frac{z^2}{4\pi}} f_{\eta_3}(x)\,dx$

$$= \mathbb{E}\ e^{\left(\frac{iz}{2}\sqrt{A_{\eta_3}}\right)^2} = \mathbb{E} e^{\frac{iz}{\sqrt{2}} N \sqrt{A_{\eta_3}}}$$

Theorem :

- $X_{\theta_4} \sim \frac{d}{dx} \theta_4(e^{-\pi x})$
- $X_{\eta_3} \sim \eta^3(ix)$ independent

$$\Rightarrow \quad \boxed{X_{\theta_4} + X_{\eta_3} \sim 4 X_{\theta_4}}$$

proof :

$$X_{\theta_4} \sim \frac{1}{\pi} \sum_{k \geq 1} \frac{E_k}{k^2} \quad , \quad X_{\eta_3} \sim \frac{1}{\pi} \sum_{k \geq 0} \frac{E_k}{(k+\frac{1}{2})^2}$$

iid Exponentials

$$\Rightarrow \frac{1}{4} X_{\theta_4} + \frac{1}{4} X_{\eta_3} \sim \frac{1}{\pi} \sum \frac{E_k}{(2k+1)^2} + \frac{E'_k}{(2k)^2} \sim X_{\theta_4}$$

Theorem

- $X_{\theta_4} \sim \frac{d}{dx}\theta_4(e^{-\pi x})$

- $X_{\eta_1} \sim \eta^{(1x)}$

- $X_{\eta_3} \sim \eta^{3}(1x)$ independent

are related as

$$\boxed{\frac{1}{9} X_{\theta_4} + \frac{1}{12} X_{\eta_1} \sim \frac{1}{4} X_{\eta_3}}$$

Eta Products

Example: $\dfrac{\eta(z)^2}{\eta(2z)} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$ is a c.d.f.

$$= \theta_4(q)$$

. rewrite as $\prod_{n\geq 1} \dfrac{1-q^n}{1+q^n} = \prod_{n\geq 1} \tanh\left(\dfrac{n\pi x}{2}\right)$ $\qquad q = e^{-\pi x}$

$$= \Pr\{X \leq x\}$$

with $\quad X = \max_{i\geq 1}\left\{\dfrac{Z_i}{i}\right\} \qquad Z_i \sim \dfrac{\pi}{2}\operatorname{sech}^2(\pi z)$

. moreover $\quad A_{\theta_4} \sim \dfrac{d}{dc}\theta_4(q) \qquad$ (Yor, 2001)

$$\Rightarrow \qquad \sqrt{A_{\theta_4}}\,.\,N = Z \sim \frac{\pi}{2}\operatorname{sech}^2(\pi z)$$

Eta Products: another example

- $$\frac{\eta(2iz)\,\eta^2(3iz)}{\eta(iz)\,\eta(6iz)} = \sum_{n \equiv 1 \bmod 6} e^{-\frac{\pi}{12} n^2 z}$$

$$\Rightarrow \quad f_X(z) = \frac{3}{4\pi} \frac{\eta(2iz)\,\eta^2(3iz)}{\eta(iz)\,\eta(6iz)} \quad \text{is a p.d.f.}$$

- the moments are computed as

$$\int_0^\infty x^p f_X(x)\,dx = \frac{1}{4.3^p} \cdot \frac{p!}{\pi^{p+2}} \left[\zeta\left(2p+2, \tfrac{1}{6}\right) + \zeta\left(2p+2, \tfrac{5}{6}\right) \right]$$

- the moment generating function

$$\mathbb{E}\, e^{zX} = \frac{1}{\sqrt{\frac{\pi z}{3}}} \frac{\tanh \sqrt{\frac{\pi z}{3}}}{1 + 3 \tanh^2 \sqrt{\frac{\pi z}{3}}}$$

Questions:

- which eta products

$$\frac{\prod_{k=1}^{n} \eta(\iota a_k)^{\alpha_k}}{\prod_{\ell=1}^{m} \eta(\iota b_\ell)^{\beta_\ell}}$$

are
- p.d.f s ?
- cdf s ?

- consequences of modularity ?
- links between random variables ?
- moments, cumulants ?

- Thank you.