

On b -ary Binomial Coefficients

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Outlines

1 Digital Binomial Identity & $\binom{n}{k}_b$

- Binary expansion
- b -ary binomial coefficients $\binom{n}{k}_b$
- Some properties

2 Negative Entries

- Binomial coefficients with negative entries
- Generalization of b -ary binomial coefficients
- Results and CONJECTURES

Binary expansion

Example

$$10 = 8 + 2 = 2^3 + 2^1 = (1010)_2$$

n	0	1	2	3	4	5	6
$(n)_2$	$(000)_2$	$(001)_2$	$(010)_2$	$(011)_2$	$(100)_2$	$(101)_2$	$(110)_2$

$$(X + Y)^6 = \sum_{k=0}^6 \binom{6}{k} X^k Y^{6-k}$$

Definition

$S_2(n) = \#$ of 1's in the binary expansion of n .

n	0	1	2	3	4	5	6
$(n)_2$	$(000)_2$	$(001)_2$	$(010)_2$	$(011)_2$	$(100)_2$	$(101)_2$	$(110)_2$
$S_2(n)$	0	1	1	2	1	2	2

Digital binomial identity

n	0	1	2	3	4	5	6
$(n)_2$	$(000)_2$	$(001)_2$	$(010)_2$	$(011)_2$	$(100)_2$	$(101)_2$	$(110)_2$
$S_2(n)$	0	1	1	2	1	2	2

$$(X + Y)^{S_2(6)} = (X + Y)^2 = X^2 Y^0 + XY + XY + X^0 Y^2$$

$$(X + Y)^{S_2(6)} = \sum_{k=0,2,4,6} X^{S_2(k)} Y^{S_2(6-k)}$$

Remark ("Carry")

$k + (n - k)$	0 + 6	1 + 5	2 + 4	3 + 3	4 + 2	5 + 1	6 + 0
	000	001	010	011	100	101	110
Binary	110	101	100	011	010	001	000
	<u>110</u>						
Carry-free	✓	✗	✓	✗	✓	✗	✓

Theorem

$$(X + Y)^{S_2(n)} = \sum_{0 \leq k \leq n} X^{S_2(k)} Y^{S_2(n-k)},$$

$$0 \leq k \lesssim_2 n = \begin{cases} 0 \leq k \leq n \\ k + (n - k) \text{ carry free} \end{cases}$$

b -ary

Definition

$S_b(n)$ = sum of all digits of n in its expansion of base b .

Example

$$6 = (110)_2 \Rightarrow S_2(6) = 2$$

$$6 = (12)_4 \Rightarrow S_4(6) = 1 + 2 = 3.$$

n	0	1	2	3	4	5	6
$(n)_4$	$(00)_4$	$(01)_4$	$(02)_4$	$(03)_4$	$(10)_4$	$(11)_4$	$(12)_4$
$S_4(n)$	0	1	2	3	1	2	3

$$(X + Y)^{S_4(6)} = (X + Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$$

Remark

Only summing over $0 \leq k \lesssim_4 6$ is not enough.

b -ary digital binomial identity

Question

$$(X + Y)^{S_2(n)} = \sum_{0 \leq k \lesssim_2 n} X^{S_2(k)} Y^{S_2(n-k)}$$

↓

$$(X + Y)^{S_b(n)} = \sum_{0 \leq k \lesssim_b n} ??? X^{S_b(k)} Y^{S_b(n-k)}$$

Lemma (L. J. C. Vignat)

$$(X + Y)^{S_b(n)} = \sum_{0 \leq k \lesssim_b n} \left(\prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} \right) X^{S_b(k)} Y^{S_b(n-k)},$$

where

$$S_b^{(j)}(m) = \# \text{ of } j\text{'s in } m\text{'s } b\text{-ary expansion.}$$

b -ary binomial coefficients

Definition

$$\binom{n}{k}_b := \begin{cases} \prod_{j=0}^{b-1} (j!)^{S_b^{(j)}(n) - S_b^{(j)}(k) - S_b^{(j)}(n-k)} & \text{if carry-free} \\ 0 & \text{otherwise} \end{cases}$$

Remark

$$(X + Y)^{S_b(n)} = \sum_{k=0}^n \binom{n}{k}_b X^{S_b(k)} Y^{S_b(n-k)}.$$

Theorem (L. J. C. Vignat)

$$\binom{n}{k}_b = \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell}, \quad \begin{cases} n = (n_{N-1} \cdots n_0)_b \\ k = (k_{N-1} \cdots k_0)_b \end{cases}$$

"carry-free" $\Leftrightarrow k_\ell \leq n_\ell,$
 $\ell = 0, \dots, N-1.$

Generating Function

Definition

The generating function of the b -ary binomial coefficients is defined as

$$f(n, b, x) := \sum_{k=0}^n \binom{n}{k}_b x^k.$$

Example ($b = 4$)

n	$f(n, 4, x)$							
1	$1 + x$	3	$(1 + x)^3$	5	$(1 + x)(1 + x^4)$	7	$(1 + x)^3(1 + x^4)$	
2	$(1 + x)^2$	4	$1 + x^4$	6	$(1 + x)^2(1 + x^4)$	8	$(1 + x^4)^2$	

Theorem (L. J. C. Vignat)

Let $n = (n_{N-1} \cdots n_0)_b$. Then,

$$f(n, b, x) = \sum_{k=0}^n \binom{n}{k}_b x^k = \prod_{\ell=0}^{N-1} (1 + x^{b^\ell})^{n_\ell}.$$

Lucas' Theorem

Theorem (Lucas)

For a prime p ,

$$\binom{n}{k} \equiv \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell} = \binom{n}{k}_p \pmod{p}$$

Proof.

A simple proof is obtained by noting that

$$\sum_{k=0}^n \binom{n}{k}_p x^k = \prod_{\ell=0}^{N-1} \left(1 + x^{p^\ell}\right)^{n_\ell} \equiv (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k, \pmod{p}$$

when expanding

$$n = \sum_{\ell=0}^{N-1} n_\ell p^\ell.$$

Pascal-like triangles

Example ($b = 4$)

Fact

$$\binom{n}{k}_b = \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell} = \binom{n_{N-1}}{k_{N-1}} \prod_{l=0}^{N-2} \binom{n_\ell}{k_\ell}.$$

Chu-Vandermonde identity

Theorem (L. J. C. Vignat)

Suppose that the addition of m and n is carry-free in base b , then

$$\binom{m+n}{r}_b = \sum_{0 \leq k \lesssim_b r} \binom{m}{k}_b \binom{n}{r-k}_b.$$

Remark

Both $m+n$ is carry free and $0 \leq k \lesssim_b n$ condition are necessary. Let $b = 2$

- ① $m = n = r = 1 \Rightarrow m + n = 2$ is not carry-free.

$$LHS = \binom{2}{1}_2 = \binom{1}{0} \binom{0}{1} = 0 \neq \sum_{0 \leq k \lesssim_2 1} \binom{1}{k}_2 \binom{1}{1-k}_2 = 2.$$

- ② The case $m = r = 2$ and $n = 1$ also shows that $0 \leq k \lesssim_b r$ cannot be replaced by $0 \leq k \leq r$.

Other properties



$$\binom{n}{k}_b = \binom{n}{n-k}_b$$

- When $\binom{n}{k}_b \neq 0$,

$$\binom{n}{k}_b = \binom{n-1}{k-1}_b + \binom{n-1}{k}_b$$

and

$$\binom{n}{k}_b = \binom{n-b^j}{k-b^j}_b + \binom{n-b^j}{k}_b$$



$$\sum_{k=0}^n \binom{n}{k}_b \binom{k}{j}_b (-1)^{S_b(k)+S_b(j)} = \delta_{n,j} = \begin{cases} 1, & n=j; \\ 0, & \text{otherwise.} \end{cases}$$

Binomial Coefficients

For $\alpha \in \mathbb{C}$,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} \implies (1 + x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

Example.

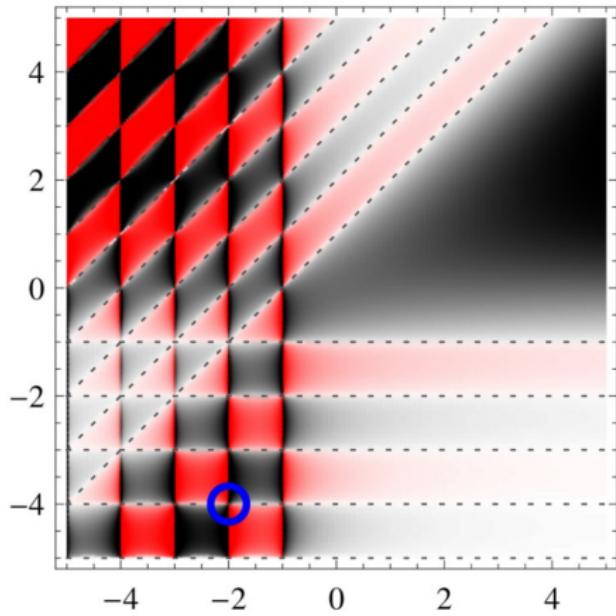
$$\binom{-2}{4} = \frac{(-2)(-3)(-4)(-5)}{4!} = 5.$$

Question:

$$\binom{-2}{-4} = ?$$

Answer:

$$\binom{x}{y} := \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(x + 1 + \varepsilon)}{\Gamma(y + 1 + \varepsilon) \Gamma(x - y + 1 + \varepsilon)} \quad [n! = \Gamma(n + 1)].$$



This scale is also visible along the line $y = 1$.

This is a plot of:

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Defined and smooth on $\mathbb{R} \setminus \{x = -1, -2, \dots\}$.

Directional limits exist at integer points:

$$\lim_{\varepsilon \rightarrow 0} \binom{-2 + \varepsilon}{-4 + r\varepsilon} = \frac{1}{2!} \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(-1 + \varepsilon)}{\Gamma(-3 + r\varepsilon)} = 3r$$

since $\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \frac{1}{\varepsilon} + O(1)$

DEF For all $x, y \in \mathbb{Z}$:

$$\binom{x}{y} := \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(x+1+\varepsilon)}{\Gamma(y+1+\varepsilon)\Gamma(x-y+1+\varepsilon)}$$

Daniel E. Loeb, Sets with a negative number of elements, Adv. Math. 91 (1992), 64–74.

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Re: Copyright

AS Armin Straub <straub@southalabama.edu>
Thu 2019-09-12 15:00
Lin Jiu

Dear Lin:

Sure, feel free to use graphs/pictures that I created. For the pictures of people, I was (mostly) using pictures I found online, so you would be stealing from the internet (like me).

If you plan to steal entire slides, then please include some credit :D

Best wishes,
Armin

On Thu, Sep 12, 2019, at 12:54 PM, Lin wrote:
> Dear Armin,
>
>
> May I steal some slides/pics/graphs from your slides of OPSFA?
>
>
> Best,
>
> Lin

Negative entries

Theorem (D. E. Loeb)

For $n \in \mathbb{Z}$,

$$\binom{n}{k} := [x^k] (1+x)^n,$$

where, when k is a negative integer, it is the coefficient of x^k of the inverse power series of $(1+x)^n$. Namely, letting $n \in \mathbb{N}$,

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{j=n}^{\infty} \binom{-n}{-j} \frac{1}{x^j}.$$

- ① If $n \geq k \geq 0$, then $\binom{n}{k} = \binom{n}{k}$.
- ② If $k \geq 0 > n$, then $\binom{n}{k} = (-1)^k \binom{n+k-1}{k}$.
- ③ If $0 > n > k$, then $\binom{n}{k} = (-1)^{n+k} \binom{-k-1}{n-k}$.
- ④ If $k > n \geq 0$, or $0 > k > n$, or $k \geq 0 > n$, then $\binom{n}{k} = 0$.

Example

Let $n = 2$.

$$\frac{1}{1+x} = \frac{1}{x} \cdot \frac{1}{1+\frac{1}{x}} = \frac{1}{x} \sum_{j=0}^{\infty} \frac{(-1)^j}{x^j} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{x^j}$$

$$\Rightarrow \frac{1}{(1+x)^2} = \frac{d}{dx} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{x^j} \right) = \sum_{j=2}^{\infty} \frac{(-1)^{j-1} j}{x^{j+1}}$$

This shows that

$$\binom{-2}{-4} = (-1)^{3-1} \cdot 3 = 3.$$

$$\binom{-2}{-4} = (-1)^{-2-4} \binom{4-1}{-2-(-4)} = \binom{3}{2} = 3.$$

b -ary

Definition

Let $n, k \in \mathbb{Z}$ with $|n| = (n_{N-1} \cdots n_1 n_0)_b$ and $|k| = (k_{N-1} \cdots k_1 k_0)_b$.

- Define $S_b(n) = \text{sign}(n)S_b(|n|)$.
- The three types of b -ary binomial coefficients are defined as

$$\binom{n}{k}_b^{(1)} := \prod_{\ell=0}^{N-1} \binom{\text{sign}(n)n_\ell}{k_\ell},$$

$$\binom{n}{k}_b^{(2)} := [x^k] \prod_{\ell=0}^{N-1} \left(1 + x^{b^\ell}\right)^{\text{sign}(n)n_\ell}$$

$$\binom{n}{k}_b^{(3)} := \left[X^{S_b(n)-S_b(k)} Y^{S_b(k)} \right] (X + Y)^{S_b(n)}$$

Remark

In SageMath, $(-7)_2 = ((-1)(-1)(-1))_2$.

Explicit expressions $n \in \mathbb{N}$

$$\binom{-n}{k}_b^{(1)} := \prod_{l=0}^{N-1} \binom{-n_l}{k_l}, \quad \binom{-n}{k}_b^{(2)} := [x^k] \prod_{l=0}^{N-1} \left(1 + x^{b^l}\right)^{-n_l}, \quad \binom{-n}{k}_b^{(3)} := [X^{-S_b(n) - S_b(k)} Y^{S_b(k)}] (X + Y)^{-S_b(n)}$$

Proposition (L. J. D. Shi)

$$\binom{-n}{k}_b^{(2)} = \begin{cases} \sum_{(j_0, \dots, j_{N-1}) \in \mathcal{P}_N(k, \mathbf{b}_N)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{j_\ell}, & \text{if } k \geq 0; \\ \sum_{(j_0, \dots, j_{N-1}) \in \mathcal{P}_N^*(-k, \mathbf{b}_N)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{-j_\ell}, & \text{if } k < 0, \end{cases}$$

$$\binom{-n}{k}_b^{(3)} := \begin{cases} \sum_{j_0 + \dots + j_{N-1} = S_b(k)} \prod_{\ell=0}^{N-1} \binom{-n_\ell}{j_\ell}, & \text{if } k \geq 0; \\ \sum_{j_0 + \dots + j_{N-1} = -S_b(k)}^* \prod_{\ell=0}^{N-1} \binom{-n_\ell}{-j_\ell}, & \text{if } k < 0, \end{cases}$$

Explicit expressions $n \in \mathbb{N}$

$$\sum_{(j_0, \dots, j_{N-1}) \in \mathcal{P}(k, \mathbf{b}_N)}, \quad \sum_{(j_0, \dots, j_{N-1}) \in \mathcal{P}^*(-k, \mathbf{b}_N)}, \quad \sum_{j_0 + \dots + j_{N-1} = S_b(k)}, \quad \sum_{j_0 + \dots + j_{N-1} = -S_b(k)}^*$$

- ① $\sum_{j_0 + \dots + j_{N-1} = S_b(k)}$ is summing over *nonnegative* integers j_0, \dots, j_{N-1} ;
- ② $\sum_{j_0 + \dots + j_{N-1} = S_b(k)}^*$ is summing over *positive* integers j_0, j_1, \dots, j_{N-1} ;
- ③ $\mathbf{b}_N := \{1, b, \dots, b^{N-1}\}$ and $\mathcal{P}(k, \mathbf{b}_N)$ is the set of *restricted partitions* of k into parts in \mathbf{b}_N , i.e., N -tuples of nonnegative integers (j_0, \dots, j_{N-1}) such that

$$j_0 b^0 + j_1 b^1 + \dots + j_{N-1} b^{N-1} = k;$$

- ④ $\mathcal{P}^*(-k, \mathbf{b}_N) = \mathcal{P}(-k, \mathbf{b}_N) \cap \mathbb{N}^N$ (N -tuples with positive component).

Example: $b = 4$, $n = -6$, $|n| = (12)_4$, $f(x) = \frac{1}{(1+x)^2(1+x^4)}$

① $k = 7 = (13)_4$, $S_4(k) = 4$.

$$\bullet \binom{-6}{7}_4^{(1)} = \binom{-1}{1} \binom{-2}{3} = \frac{(-1)}{1} \cdot \frac{(-2)(-3)(-4)}{3!} = (-1) \cdot (-4) = 4;$$

$$\bullet f(x) = 1 - 2x + \cdots + 4x^6 - 4x^7 + O(x^8). \Rightarrow \binom{-6}{7}_4^{(2)} = -4;$$

$$\bullet (X+Y)^{-3} = X^{-3} \left(1 - \cdots + 15 \frac{Y^4}{X^4} - O\left(\frac{Y^5}{X^5}\right) \right) \Rightarrow \binom{-6}{7}_4^{(3)} = 15.$$

② $k = -8$, $8 = (20)_4$, $S_4(-8) = -2$

$$\bullet \binom{-6}{-8}_4^{(1)} = \binom{-1}{-2} \binom{-2}{0} = (-1) \cdot 1 = -1;$$

$$\bullet f(x) = \frac{1}{x^6} - \frac{2}{x^7} + \frac{3}{x^8} + O\left(\frac{1}{x^9}\right). \Rightarrow \binom{-6}{-8}_4^{(2)} = 3;$$

$$\bullet (X+Y)^{-3} = X^{-3} \left(1 + \frac{Y}{X} \right)^{-3} = X^{-3} \left(\frac{X^3}{Y^3} - \cdots \right) \Rightarrow \binom{-6}{-8}_4^{(3)} = 0.$$

③ Alternatively, $\mathbf{4}_2 = \{1, 4\}$.

$$\bullet 7 = 1 \cdot 4 + 3 = 0 \cdot 4 + 7 \Rightarrow \mathcal{P}(7, \mathbf{4}_2) = \{(1, 3), (0, 7)\}.$$

$$\binom{-6}{7}_4^{(2)} = \binom{-1}{1} \binom{-2}{3} + \binom{-1}{0} \binom{-2}{7} = (-1) \cdot (-4) + 1 \cdot (-8) = -4.$$

$$\bullet \{(j_0, j_1) \in \mathbb{N}^2 : j_0 + j_1 = 2\} = \{(1, 1)\} \Rightarrow \binom{-6}{8}_4^{(3)} = \binom{-1}{1} \binom{-2}{7} = 1 \cdot 0 = 0.$$

Proof and Explanation

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{j=1}^{\infty} \binom{-n}{-j} \frac{1}{x^j}.$$

From the generating function

$$\sum_{k=0}^{\infty} \binom{-n}{k}_b^{(2)} x^k = \prod_{\ell=0}^{N-1} \left(\frac{1}{1+x^{b^\ell}} \right)^{n_\ell} = \prod_{\ell=0}^{N-1} \left(\sum_{j_\ell=0}^{\infty} \binom{-n_\ell}{j_\ell} x^{j_\ell b^\ell} \right).$$

When, $n > 0$, each factor is $\left(1 + x^{b^\ell}\right)^{n_\ell}$, which restricts each $j_\ell \in [0, n_\ell] \subset [0, b-1]$.

$$j_0 b^0 + j_1 b^1 + \cdots + j_{N-1} b^{N-1} = k \Rightarrow j_\ell = k_\ell$$

Thus,

$$\binom{n}{k}_b = \prod_{\ell=0}^{N-1} \binom{n_\ell}{k_\ell} = [x^k] \prod_{\ell=0}^{N-1} \left(1 + x^{b^\ell}\right)^{n_\ell}.$$

The case for $(X+Y)^{S_b(n)}$ is similar.

Lucas' theorem

Recall

$$\sum_{k=0}^n \binom{n}{k}_p x^k = \prod_{\ell=0}^{N-1} (1 + x^{p^\ell})^{n_\ell} \equiv (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \pmod{p}$$

Taking the reciprocal indicates the following.

Proposition (L. J. D. Shi)

For $n, k \in \mathbb{Z}$ and a prime p ,

$$\binom{n}{k} \equiv \binom{n}{k}_p^{(2)} \pmod{p}.$$

Neither $\binom{n}{k}_p^{(3)}$ nor $\binom{n}{k}_p^{(1)}$ satisfies this congruence.

Remark

It is obvious that $\binom{n}{k}_p^{(1)}$ has only finitely many non-zero terms, for fixed n .

Pascal's triangle

SageMath

Conjecture (L. J. D. Shi)

For the following recurrence

$$\binom{n}{k}_b^{(j)} + \binom{n}{k-1}_b^{(j)} = \binom{n+1}{k}_b^{(j)},$$

- ① when $j = 2$, it holds $\forall n, b \in \mathbb{N}$ and $\forall k \in \mathbb{Z}$;
- ② when $j = 1$, it holds if k has more digits than n , or k is nonnegative and $b \nmid k$, or if k is negative and $k \not\equiv 1 \pmod{b}$;
- ③ when $j = 3$, it hold if k is nonnegative and $b \nmid k$, or if k is negative and $k \not\equiv 1 \pmod{b}$.

Symmetry

Conjecture (~2019-09-05)

$$\binom{n}{k}_b^{(2)} = \binom{n}{n-k}_b^{(2)}$$

For $n, k \in \mathbb{N}$,

$$\binom{-n}{k}_b^{(2)} = \binom{-n}{-n-k}_b^{(2)} \quad \text{and} \quad \binom{-n}{-k}_b^{(2)} = \binom{-n}{-n+k}_b^{(2)}$$

Proposition (L. J, D. Shi 2019-09-15/16)

$$\binom{n}{k}_b^{(2)} = \binom{n}{n-k}_b^{(2)}.$$

Symmetry

For $n, k \in \mathbb{N}$,

$$\binom{-n}{k}_b^{(2)} = \binom{-n}{-n-k}_b^{(2)} \quad \text{and} \quad \binom{-n}{-k}_b^{(2)} = \binom{-n}{-n+k}_b^{(2)}$$

$$\sum_{k=0}^{\infty} \binom{-n}{k}_b^{(2)} x^k = \underbrace{\prod_{\ell=0}^{N-1} \left(1 + x^{b^\ell}\right)^{-n_\ell}}_{f(x)=} = \sum_{k=n}^{\infty} \binom{-n}{-k}_b^{(2)} \frac{1}{x^k}.$$

$$f\left(\frac{1}{x}\right) = \prod_{\ell=0}^{N-1} \frac{1}{\left(1 + \left(\frac{1}{x}\right)^{b^\ell}\right)^{n_\ell}} = \prod_{\ell=0}^{N-1} \frac{x^{n_\ell b^\ell}}{\left(x^{b^\ell} + 1\right)^{n_\ell}} = x^n f(x).$$

$$(FG)^{(m)} = \sum_{j=0}^m \binom{m}{j} F^{(j)} G^{(m-j)}.$$

Chu-Vandermonde identity

SageMath

Somehow, it fails.....

More elementary properties?

As mentioned before, $\binom{n}{k}_b^{(1)}$, for fixed n , has only finitely many nonzero values for $k \in \mathbb{Z}$.

Example

Let $b = 3$. $n = -4, 4 = (11)_3$.

k	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
	1	-1	-1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1

What is the sum $(\text{mod } 3) = 1$

See SageMath

It seems that for any $n \in \mathbb{N}$,

$$\sum_{k \in \mathbb{Z}} \binom{-n}{k}_3^{(1)} \not\equiv -1 \pmod{3}.$$

$$\sum_{k=0}^n \binom{n}{k}_3 \equiv \sum_{k=0}^n \binom{n}{k} = 2^n \not\equiv 0 \pmod{3}.$$

Restricted partition

Definition

Given a vector $\mathbf{d} := (d_1, \dots, d_m)$ of positive integers. Let

$$W(s, \mathbf{d}) = |\{(x_1, \dots, x_n) \in \mathbb{N}^n : d_1x_1 + d_2x_2 + \dots + d_mx_m = s\}|.$$

$$F(t, \mathbf{d}) = \prod_{j=1}^m \frac{1}{1 - t^{d_j}} = \sum_{s=0}^{\infty} W(s, \mathbf{d}) t^s.$$

$\mathbf{b}_N := \{1, b, \dots, b^{N-1}\}$ and $\mathcal{P}(k, \mathbf{b}_N)$ is the set of *restricted partitions* of k into parts in \mathbf{b}_N , i.e., N -tuples of nonnegative integers j_0, \dots, j_{N-1} such that $j_0 b^0 + j_1 b^1 + \dots + j_{N-1} b^{N-1} = k$;

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- [1] K. Dilcher and C. Vignat, An explicit form of the polynomial part of a restricted partition function, Research in Number Theory 2017, 3:1.