

INTRODUCTION TO FOUR SYMBOLIC INTEGRATION METHODS: TWO EXAMPLES

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1. INTRODUCTION

The following two examples will be used.

Example 1.1.

$$I := \int_0^{\infty} \frac{dx}{1+x^2} = \tan^{-1}(x) \Big|_{x=0}^{x=\infty} = \frac{\pi}{2}.$$

Example 1.2.

$$I' := \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

2. THE METHOD OF BRACKETS (MoB)

MoB evaluates the definite integral

$$\int_0^{\infty} f(x) dx$$

(most of the time) in terms of **SERIES**, with *ONLY SEVEN* rules.

Definition 2.1. The *indicator* is defined by

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}.$$

Also, their product is denoted by

$$\phi_{1,\dots,r} := \phi_{n_1,\dots,n_r} = \phi_{n_1} \phi_{n_2} \cdots \phi_{n_r} = \prod_{i=1}^r \phi_{n_i}.$$

Rules

P_1 : If

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1},$$

then,

$$\int_0^{\infty} f(x) dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$$

which is called the bracket series;

P_2 : For $\alpha < 0$,

$$(a_1 + \cdots + a_r)^\alpha \mapsto \sum_{n_1,\dots,n_r} \phi_{1,\dots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)};$$

P_3 : For each bracket series, we assign index = # of sums - # of brackets;

E_1 :

$$\sum_n \phi_n F(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} F(n^*) F(-n^*),$$

where n^* solves $\alpha n + \beta = 0$;

E_2 :

$$\begin{aligned} & \sum_{n_1, \dots, n_r} \phi_{1, \dots, r} F(n_1, \dots, n_r) \prod_{i=1}^r \langle a_{i1}n_1 + \dots + a_{ir}n_r + c_i \rangle \\ &= \frac{F(n_1^*, \dots, n_r^*) \prod_{i=1}^r \Gamma(-n_i^*)}{|\det A|}, \end{aligned}$$

where

$$(n_1^*, \dots, n_r^*) \text{ solves } \begin{cases} a_{11}n_1 + \dots + a_{1r}n_r + c_1 = 0 \\ \dots & \dots ; \\ a_{r1}n_1 + \dots + a_{rr}n_r + c_r = 0 \end{cases}$$

E_3 : The value of a multi-dimensional bracket series of **POSITIVE** index is obtained by computing all the contributions of maximal rank by Rule E_2 . These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

E_4 : Let $k \in \mathbb{N}$ be fixed. In the evaluation of series, the rule

$$(-km)_{-m} = \frac{k}{k+1} \cdot \frac{(-1)^m (km)!}{((k+1)m)!}$$

must be used to eliminate Pochhammer symbols with negative index and negative integer base.

Example 2.2. We let

$$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \phi_n \Gamma(n+1) x^{2n}.$$

(1) By P_1 ,

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \int_0^{\infty} f(x) dx = \sum_{n=0}^{\infty} \phi_n \Gamma(n+1) \langle 2n+1 \rangle.$$

(2) To apply E_1 , we solve for n^* :

$$2n^* + 1 = 0 \implies n^* = -\frac{1}{2}.$$

By E_1 ,

$$I = \frac{1}{|2|} \Gamma(n^* + 1) \Gamma(-n^*) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)^2 = \frac{1}{2} (\sqrt{\pi})^2 = \frac{\pi}{2}.$$

Alternatively, we can first apply P_2 to have

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = \sum_{n_1, n_2} \phi_{n_1} \phi_{n_2} 1^{n_1} x^{2n_2} \frac{\langle 1+n_1+n_2 \rangle}{\Gamma(1)} = \sum_{1,2} \phi_{1,2} x^{2n_2} \langle 1+n_1+n_2 \rangle.$$

Then,

$$I = \int_0^\infty \sum_{1,2} \phi_{1,2} x^{2n_2} \langle 1+n_1+n_2 \rangle dx = \sum_{1,2} \phi_{1,2} \langle n_1+n_2+1 \rangle \langle 2n_2+1 \rangle.$$

Solving the linear system:

$$\left. \begin{array}{l} n_1 + n_2 = -1 \\ 2n_2 = -1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = 2 \\ (n_1^*, n_2^*) = (-\frac{1}{2}, -\frac{1}{2}) \end{array} \right. .$$

By E_2 ,

$$I = \frac{1}{|2|} \Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{2}.$$

Example 2.3. We let

$$f(x) = \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+1 \rangle.$$

Then,

$$I' = \frac{1}{|2|} \frac{\Gamma(-\frac{1}{2}+1)}{\Gamma(2(-\frac{1}{2})+2)} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{2}.$$

3. NEGATIVE DIMENSION INTEGRATION METHOD

Example 3.1. Consider for $\alpha > 0$,

$$J(\alpha) := \int_{\mathbb{R}} e^{-\alpha(1+x^2)} dx.$$

- By the Gaussian distribution,

$$1 = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Let $\mu = 0$ and $1/(2\sigma^2) = \alpha$. We have

$$(3.1) \quad J(\alpha) = e^{-\alpha} \int_{\mathbb{R}} e^{-\alpha x^2} dx = e^{-\alpha} \sqrt{\frac{\pi}{\alpha}} = \sqrt{\pi} \sum_{n=0}^{\infty} \phi_n \alpha^{n-\frac{1}{2}}.$$

- On the other hand, assuming that we can interchange the integration and summation, we have

$$(3.2) \quad J(\alpha) = \int_{\mathbb{R}} \sum_{m=0}^{\infty} \phi_m \alpha^m (1+x^2)^m dx = \sum_{m=0}^{\infty} \phi_m \left(\int_{\mathbb{R}} (1+x^2)^m dx \right) \alpha^m.$$

Now, define

$$I_m := \int_{\mathbb{R}} (1+x^2)^m dx.$$

To match (3.1) and (3.2), we see $n - 1/2 = m$, i.e., $n = m + 1/2$. Comparing coefficients indicates

$$\begin{aligned}\sqrt{\pi}\phi_{m+\frac{1}{2}} = \phi_m I_m &\implies I_m = \sqrt{\pi} \frac{\phi_{m+\frac{1}{2}}}{\phi_m} \\ &= \sqrt{\pi} \frac{(-1)^{m+\frac{1}{2}}}{(-1)^m} \cdot \frac{\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} \\ &= \frac{\sqrt{\pi}i}{(m+1)_{\frac{1}{2}}}.\end{aligned}$$

Thus, by

$$(-a)_n = (-1)^n (a - n + 1)_n,$$

we have

$$I_m = \frac{\sqrt{\pi}i}{(-1)^{\frac{1}{2}} (-m - \frac{1}{2})_{\frac{1}{2}}} = \frac{\sqrt{\pi}}{(-m - \frac{1}{2})_{\frac{1}{2}}},$$

so that

$$I = \frac{I_{-1}}{2} = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{(\frac{1}{2})_{\frac{1}{2}}} = \frac{1}{2} \cdot \frac{\sqrt{\pi}\Gamma(\frac{1}{2})}{\Gamma(1)} = \frac{\pi}{2}.$$

4. INTEGRATION BY DIFFERENTIATION

Theorem 4.1 (A. Kempf, D. Jackson, and A. Morales). *Let $\partial_\varepsilon = \partial/\partial\varepsilon$.*

$$\begin{aligned}\int_a^b f(x) dx &= \lim_{\varepsilon \rightarrow 0} f(\partial_\varepsilon) \frac{e^{\varepsilon b} - e^{\varepsilon a}}{\varepsilon}, \\ \int_{-\infty}^{\infty} f(x) dx &= \lim_{\varepsilon \rightarrow 0} 2\pi f(-i\partial_\varepsilon) \delta(\varepsilon) = 2\pi \delta(i\partial_\varepsilon) f(\varepsilon), \\ \int_0^{\infty} f(x) dx &= \lim_{\varepsilon \rightarrow 0} f(-\partial_\varepsilon) \frac{1}{\varepsilon}, \\ \int_{-\infty}^0 f(x) dx &= \lim_{\varepsilon \rightarrow 0} f(\partial_\varepsilon) \frac{1}{\varepsilon}, \\ \int_{-\infty}^{\infty} f(x) dx &= \lim_{\varepsilon \rightarrow 0} [f(-\partial_\varepsilon) + f(\partial_\varepsilon)] \frac{1}{\varepsilon},\end{aligned}$$

where $i^2 = 1$ and δ is the Dirac-delta function.

Example 4.2. Rewrite

$$\begin{aligned}f(x) &= \frac{\sin x}{x} = \frac{1}{x} \cdot \frac{1}{2i} (e^{ix} - e^{-ix}) \\ I' &= \lim_{\varepsilon \rightarrow 0} f(-\partial_\varepsilon) \frac{1}{\varepsilon} = \frac{1}{2i} \lim_{\varepsilon \rightarrow 0} (e^{-i\partial_\varepsilon} - e^{i\partial_\varepsilon}) \frac{1}{\partial_\varepsilon} \circ \frac{1}{\varepsilon}.\end{aligned}$$

Note that $1/\partial_\varepsilon$ is the inverse operation of derivative, i.e., integration.

$$I' = \frac{1}{2i} \lim_{\varepsilon \rightarrow 0} (e^{-i\partial_\varepsilon} - e^{i\partial_\varepsilon}) \circ (\ln \varepsilon + c)$$

Recall that for the derivative operator ∂_ε ,

$$e^{a\partial_\varepsilon} \circ f(\varepsilon) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \partial_\varepsilon^n \circ f(\varepsilon) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(\varepsilon) = f(\varepsilon + a).$$

Therefore,

$$\begin{aligned} I' &= \frac{1}{2i} \lim_{\varepsilon \rightarrow 0} [(\ln(\varepsilon - i) + c) - (\ln(\varepsilon + i) + c)] \\ &= \frac{1}{2i} \lim_{\varepsilon \rightarrow 0} [\ln(\varepsilon - i) - \ln(\varepsilon + i)] = \frac{1}{2i} \left(\frac{-i\pi}{2} - \frac{i\pi}{2} \right) = \frac{\pi}{2}. \end{aligned}$$

5. HYPERGEOMETRIC FORM

Proposition 5.1 (P. Blaschke). *Let f be holomorphic near the origin, $\alpha \neq -1$, $\beta \neq 0$ and $-(\alpha + 1)/\beta \notin \mathbb{N}$. Then,*

$$\int x^\alpha f(x^\beta) dx = \frac{x^{\alpha+1}}{\alpha+1} f \left(\frac{\alpha+1}{\beta} \middle| x^\beta \right) + C,$$

where

$$f \left(\frac{a}{c} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \cdot \frac{f^{(n)}(0)}{n!} x^n.$$

Corollary 5.2. *For $p \leq q + 1$,*

$$\int x^\alpha {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ c_1, \dots, c_q \end{matrix} \middle| \gamma x^\beta \right) dx = \frac{x^{\alpha+1}}{\alpha+1} {}_{p+1}F_{q+1} \left(\begin{matrix} a_1, \dots, a_p, \frac{\alpha+1}{\beta} \\ c_1, \dots, c_q, 1 + \frac{\alpha+1}{\beta} \end{matrix} \middle| \gamma x^\beta \right) + C.$$

Example 5.3. Let

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,$$

we see $f^{(n)}(0) = (-1)^n n!$. Thus,

$$\begin{aligned} \int f(x^2) dx &= x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \cdot \frac{(-1)^n n!}{n!} x^{2n} \\ &= x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1)_n}{\left(\frac{3}{2}\right)_n} \cdot \frac{(-x^2)^n}{n!} \\ &= {}_2F_1 \left(\begin{matrix} 1, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| -x^2 \right) \\ &= \tan^{-1}(x). \end{aligned}$$

Thus,

$$I = \tan^{-1}(x) \Big|_{x=0}^{x=\infty} = \frac{\pi}{2}.$$

Example 5.4. Note that

$$\begin{aligned} {}_0F_1\left(\frac{3}{2} \mid -\frac{x^2}{4}\right) &= \sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)_n} \cdot \frac{(-1)^n x^{2n}}{4^n \cdot n!} \\ &= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{\left(\frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2n+1}{2}\right) 2^n \cdot 2^n n!} \\ &= \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \frac{\sin x}{x}. \end{aligned}$$

Thus,

$$\int \frac{\sin x}{x} dx = x {}_1F_2\left(\frac{1}{2} \mid -\frac{x^2}{4} \mid \frac{3}{2}, \frac{3}{2}\right) (= \text{Si}(x)).$$

Lemma 5.5. As $z \rightarrow -\infty$,

$$(-z)^\alpha {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ c_1, \dots, c_q \end{matrix} \mid z\right) \rightarrow \prod_{\substack{l=1 \\ \alpha_l \neq \alpha}}^p \frac{\Gamma(a_l - \alpha)}{a_l} \prod_{j=1}^q \frac{\Gamma(c_j)}{\Gamma(c_j - \alpha)},$$

iff $p \geq q - 1$ and

- for $p > q - 1$, $\alpha = \min(a_1, \dots, a_p)$;
- for $p = q - 1$, $\alpha = \min(a_1, \dots, a_p) < \sigma - \frac{1}{2}$, where

$$\sigma = \sum_j c_j - \sum_l a_l.$$

Let $z = -x^2/4 \Leftrightarrow x = 2(-z)^{1/2}$, we see

$$\alpha = \frac{1}{2} = \min\left(\frac{1}{2}\right) < 2 = \left(\frac{3}{2} + \frac{3}{2} - \frac{1}{2}\right) - \frac{1}{2}.$$

Thus,

$$\lim_{x \rightarrow \infty} x {}_1F_2\left(\frac{1}{2} \mid -\frac{x^2}{4} \mid \frac{3}{2}, \frac{3}{2}\right) = 2 \left(\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}\right)^2 = \frac{\pi}{2}.$$

On the other hand,

$$\lim_{x \rightarrow 0} x {}_1F_2\left(\frac{1}{2} \mid -\frac{x^2}{4} \mid \frac{3}{2}, \frac{3}{2}\right) = 0 \cdot 1 = 0.$$

Therefore,

$$I' = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

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