The Gamma Function

Idea: Find a (reasonable) function that interpolates $(n-1)!$.

What does "reasonable" mean?

- At least smooth
- If possible defined on $\mathbb{C}$ and meromorphic
- Satisfies the properties of $(n-1)! =: f(n)$:

$$f(n+1) = n\,f(n), \quad f(1) = 1.$$

Does this define a unique function?

No! Consider $f(z) + \sin(k\pi z)$, for any $k \in \mathbb{Z}$.

Need another condition: Logarithmic convexity:

$$f(tx_1-(1-t)x_2) \le f(x_1)^t f(x_2)^{1-t}$$

Equivalent definition:

$$f''(x)f(x) \ge f'(x)^2 \quad \text{when } f \text{ is twice diff.}$$

__Bohr-Mollerup Theorem.__ There is a __unique__ function $f(x)$ defined for $x>0$ that satisfies

$$f(1)=1,$$
$$f(x+1) = x\,f(x) \quad (x>0),$$
f is logarithmically convex.

This is the gamma function $\Gamma(x)$.

__Euler Integral__ (Main definition):

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx \quad (\mathrm{Re}(z)>0)$$

All properties follow from this.

Meromorphic continuation:

Analytic everywhere on $\mathbb{C}$, except for simple poles at $z=0,-1,-2,\dots$,

with $\mathrm{Res}(\Gamma,-n) = \frac{(-1)^n}{n!}$, $n=0,1,2,\dots$

Reflection formula:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad (z\in\mathbb{Z})$$

$z=\frac{1}{2}$:

$$\Gamma(\tfrac{1}{2})^2 = \pi, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

(3)

Dirichle character

$\chi : \mathbb{Z} \to \mathbb{C}$, satisfying

1.) $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$

2.) There is an integer $k \geq 1$ such that

(a) $\chi(n) \begin{cases} = 0 & \text{if } \gcd(n,k) > 1 \\ \neq 0 & \text{if } \gcd(n,k) = 1 \end{cases}$

(b) $\chi(n+k) = \chi(n)$ for all $n \in \mathbb{Z}$

Ex.: $k = 3$

n	$\chi_0(n)$	$\chi_1(n)$
0	0	0
1	1	1
2	1	−1

$k = 4$

n	$\chi_0(n)$	$\chi_1(n)$
0	0	0
1	1	1
2	0	0
3	1	−1

$k = 5$

n	χ_0	χ_1	χ_2	χ_3
0	0	0	0	0
1	1	1	1	1
2	1	i	−1	$-i$
3	1	$-i$	−1	i
4	1	−1	1	−1

$\chi(1) = 1$

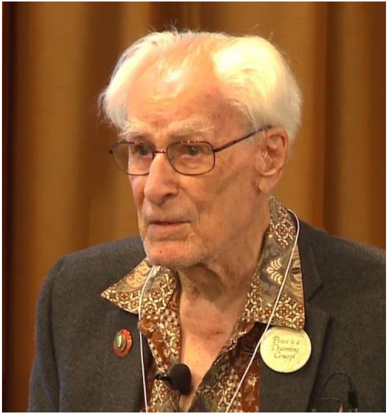
$\chi(k-1) = \pm 1$

Orthogonality

Principal character: $\chi(n) = 1$ for all n, $\gcd(n,k) = 1$

Primitive character:

Not induced by a character of smaller modulus



Happy 103rd birthday, Richard Guy

Infinite products involving Dirichlet characters and cyclotomic polynomials

Karl Dilcher

Dalhousie Number Theory Seminar, Sept. 30, 2019

Joint work with

Christophe Vignat

(Université d'Orsay and Tulane University)

1. Introduction

Well-known fact about infinite products:

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k}\right)$$

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Related: Weierstrass factorization theorem which gives, e.g.,

$$ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}=\frac{1}{\Gamma(z)}.$$

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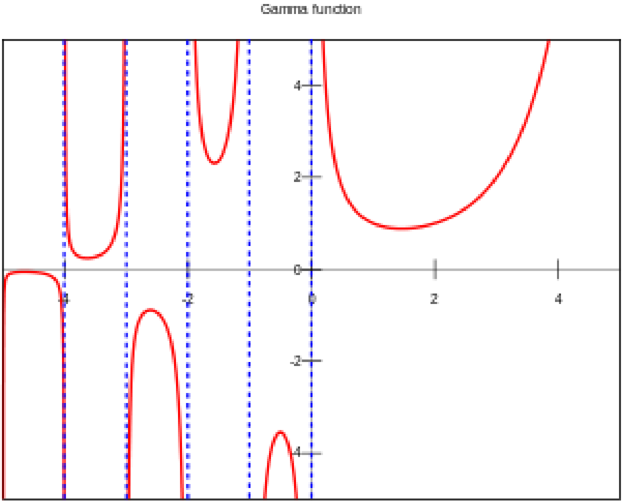
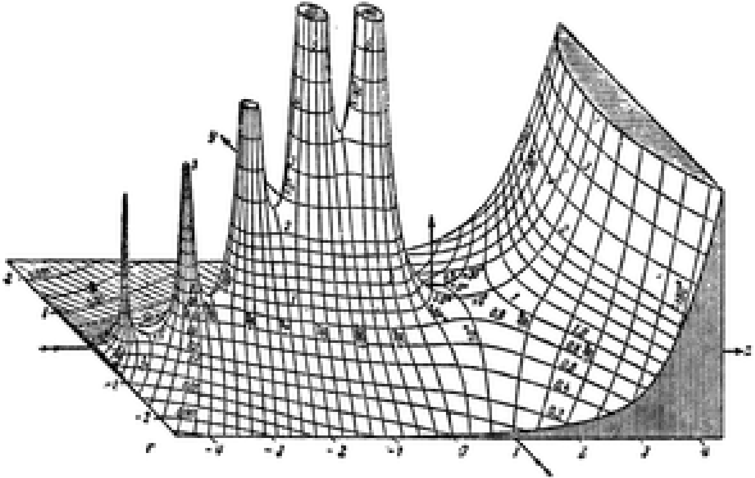
First indication that the gamma function may be involved.



Source: Wikipedia, “Gamma Function".



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PRO JUVENTUTE 1957
$e^{i\varphi} = \cos\varphi + i \sin\varphi$
LEONHARD EULER 1707-1783
5
+5 HELVETIA

A general result:

A convergent infinite product of a rational function in the index k can always be written as a product or quotient of finitely many values of the gamma function.

The last identity is an ingredient in the proof of this fact.

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A convergent infinite product of a rational function in the index k can always be written as a product or quotient of finitely many values of the gamma function.

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Goal of this talk: To extend the identity

$$\prod_{k=1}^{\infty}\left(1-\frac{(-1)^k}{2k+1}\right)=\frac{\pi}{4}\sqrt{2}$$

in a different direction.

2. Main Result

Let χ be the unique nontrivial Dirichlet character modulo 4, i.e., the periodic function of period 4 defined by $\chi(1) = 1, \chi(3) = -1$, and $\chi(0) = \chi(2) = 0$.

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and as a function of the complex variable z as

$$\prod_{k=2}^{\infty} \left(1 - \chi(k)\frac{z}{k}\right) = \frac{\sqrt{2}}{1-z} \sin \frac{(1-z)\pi}{4}. \tag{2}$$

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This is a special case of the following result.

Theorem 1

Let χ be a primitive nonprincipal Dirichlet character with conductor $q > 2$. Then

$$\prod_{k=2}^{\infty}\left(1-\chi(k)\frac{z}{k}\right) = \frac{(2\pi)^{\varphi(q)/2}}{(1-z)\epsilon(q)} \cdot \prod_{\substack{j=1\\(j,q)=1}}^{q-1} \frac{1}{\Gamma\left(\frac{j-\chi(j)z}{q}\right)},$$

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where $\epsilon(q)$ is defined by

$$\epsilon(q) = \begin{cases} \sqrt{p} & \text{when } q \text{ is a power of a prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

Main ingredients in proof:

1. Infinite product expansion for $1/\Gamma(z)$ leads to

$$\frac{\Gamma(u)}{\Gamma(u+v)} = e^{\gamma v}\prod_{k=0}^{\infty}\left(1+\frac{v}{u+k}\right)e^{-v/(k+1)},$$

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and this, in turn, gives rise to

Lemma 2

Let $n \in \mathbb{N}$, $a, z_1, \dots, z_n \in \mathbb{C}$ with $z_j \neq 0$ for $j = 1, \dots, n$, and let $f : \{1, 2, \dots, n\} \to \mathbb{C}$ satisfy $f(1) + \cdots + f(n) = 0$. Then

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$$\prod_{k=0}^{\infty} \prod_{j=1}^{n} \left(1 - f(j) \frac{a}{z_j + k}\right) = \prod_{j=1}^{n} \frac{\Gamma(z_j)}{\Gamma(z_j - f(j)a)}.$$

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Lemma 3 (Chamberland and Straub, 2013)

For any integer $n \geq 2$ and prime p we have

$$\prod_{\substack{j=1 \\ (j,n)=1}}^{n-1} \Gamma\left(\frac{j}{n}\right) = \begin{cases} (2\pi)^{\varphi(n)/2} & \text{if } n \text{ is not a prime power,} \\ \frac{1}{\sqrt{p}}(2\pi)^{\varphi(n)/2} & \text{if } n = p^{\nu}, \nu \geq 1. \end{cases}$$

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This extends the well-known identity

$$\prod_{j=1}^{n-1} \Gamma\left(\frac{j}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}.$$

Marc (left) and Armin (middle)
after being hit by a rogue wave in Peggy's Cove, Nova Scotia.

Using the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \pm 2, \ldots :$$

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Corollary 4

Let χ be an odd primitive Dirichlet character with conductor $q > 2$. Then

$$\prod_{k=2}^{\infty} \left(1 - \chi(k)\frac{z}{k}\right) = \frac{2^{\varphi(q)/2}}{(1-z)\epsilon(q)} \cdot \prod_{\substack{j=1 \\ (j,q)=1}}^{\lfloor \frac{q-1}{2} \rfloor} \sin\left(\pi \frac{j - \chi(j)z}{q}\right),$$

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and in particular,

$$\prod_{k=2}^{\infty}\left(1 - \frac{\chi(k)}{k}\right) = \frac{\pi 2^{\varphi(q)/2}}{q\epsilon(q)} \cdot \prod_{\substack{j=2\\(j,q)=1}}^{\lfloor\frac{q-1}{2}\rfloor} \sin\left(\pi\frac{j-\chi(j)}{q}\right).$$

Example 1. Let $q = 3$. Then the only nonprincipal character is given by $\chi(1) = 1$ and $\chi(2) = -1$. Then

$$\prod_{k=2}^{\infty}\left(1 - \chi(k)\frac{z}{k}\right) = \frac{2}{(1-z)\sqrt{3}} \cdot \sin(\tfrac{\pi}{3}(1-z)),$$

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and with $z = 1$ and $z = \frac{1}{2}$ we get, respectively,

$$\prod_{k=2}^{\infty}\left(1 - \frac{\chi(k)}{k}\right) = \frac{2\pi}{3\sqrt{3}}, \qquad \prod_{k=2}^{\infty}\left(1 - \frac{\chi(k)}{2k}\right) = \frac{2}{\sqrt{3}}.$$

Example 2. $q = 5$ is the smallest conductor that has nonreal characters. We choose the one (of two) that is given by $\chi(1) = 1$, $\chi(2) = i$, $\chi(3) = -i$ and $\chi(4) = -1$. Then

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and

$$\begin{aligned}\prod_{k=2}^{\infty}\left(1-\frac{\chi(k)}{k}\right) &= \frac{4\pi}{5\sqrt{5}}\cdot\sin(\tfrac{\pi}{5}(2-i)) \\ &= \frac{4\pi}{5\sqrt{5}}\left(\sin(\tfrac{2\pi}{5})\cosh(\tfrac{\pi}{5}) - i\cos(\tfrac{2\pi}{5})\sinh(\tfrac{\pi}{5})\right).\end{aligned}$$

3. Some multiple L-series

Example: Let χ_3 and χ_{-4} be the nonprincipal characters with $q = 3$ and $q = 4$, respectively. Well-known identities:

$$\sum_{k=1}^{\infty} \frac{\chi_3}{k} = \frac{\pi}{3\sqrt{3}}, \qquad \sum_{k=1}^{\infty} \frac{\chi_{-4}}{k} = \frac{\pi}{4}.$$

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More generally, let χ be a Dirichlet character with $q \geq 2$. For $n \geq 1$, consider

$$L_n(\chi) := \sum_{1 \leq k_1 < \cdots < k_n} \frac{\chi(k_1)}{k_1} \cdots \frac{\chi(k_n)}{k_n}.$$

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Expanding the infinite product, we obtain

$$\prod_{k=1}^{\infty} \left(1 - \chi(k)\frac{z}{k}\right) = 1 + \sum_{n=1}^{\infty} (-1)^n L_n(\chi) z^n.$$

Corollary 5

(a) If χ is the nonprincipal character with $q = 3$, then

$$L_{2n}(\chi) = \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}, \qquad L_{2n+1}(\chi) = \frac{(-1)^n}{(2n+1)!\sqrt{3}}\left(\frac{\pi}{3}\right)^{2n+1}.$$

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In these cases:
Only one factor on the right of our main result.

Can this be generalized?

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$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the summation is over all $j_1, j_2, \dots, j_{n-k+1} \geq 0$ satisfying

$$j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = k,$$
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Example:

$$B_{n,0}(x_1, x_2, \dots, x_{n+1}) = 0, \; B_{n,1}(x_1, x_2, \dots, x_n) = x_n, \; B_{n,n}(x_1) = x_1^n.$$

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The smallest case not belonging to these sequences is $B_{3,2}(x_1, x_2) = 3x_1 x_2$.

Important use: Faà di Bruno's formula:

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)).$$

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Applying this to the digamma function

$$\psi(z) = \Gamma'(z)/\Gamma(z), \qquad z \neq 0, -1, -2, \ldots :$$

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Lemma 6

For $n \in \mathbb{N}$ we have

$$\frac{d^n}{dx^n}\Gamma(y) = \Gamma(y) \sum_{k=1}^{n} B_{n,k}(\psi(y), \psi_1(y), \ldots, \psi_{n-k}(y)),$$

$$\frac{d^n}{dx^n}\frac{1}{\Gamma(y)} = \frac{1}{\Gamma(y)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi(y), \psi_1(y), \ldots, \psi_{n-k}(y)),$$

where $\psi_j(y) = \psi^{(j)}(y)$ and $\psi_0(y) = \psi(y)$.

Theorem 7

Let χ be a primitive nonprincipal Dirichlet character with $q > 2$. Then for $n \in \mathbb{N}$,

$$L_n(\chi)$$

$$= \frac{1}{q^n} \sum{}^{*} \prod_{j\in\Phi} \frac{\chi(j)^{k_j}}{k_j!} \sum_{k=1}^{k_j} (-1)^k B_{k_j,k}\left(\psi(\tfrac{j}{q}), \psi_1(\tfrac{j}{q}), \ldots, \psi_{k_j-k}(\tfrac{j}{q})\right),$$

with index set $\Phi := \{j \mid 1 \leq j \leq q-1, \gcd(j,q) = 1\}$, where the summation $\sum^*$ is over all $k_j (j \in \Phi)$ that sum to n.

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with index set $\Phi := \{j \mid 1 \le j \le q-1, \gcd(j,q) = 1\}$, where the summation $\sum^{*}$ is over all $k_j (j \in \Phi)$ that sum to n.

Example 1: For $n = 1$, the product reduces to a single factor. Since $B_{1,1}(x_1) = x_1$, we get the well-known identity

$$L_1(\chi) = \frac{1}{q}\sum_{j\in\Phi} \chi(j)(-1)\psi(\tfrac{j}{q}).$$

Corollary 8

Let χ be a primitive nonprincipal odd character with $q > 2$. Then

$$\sum_{1\le k<\ell} \frac{\chi(k)}{k}\frac{\chi(\ell)}{\ell}$$

$$= \frac{\pi^2}{2q^2}\left[\left(\sum_{j=1}^{\lfloor\frac{q-1}{2}\rfloor} \chi(j)\cot(\tfrac{\pi j}{q})\right)^2 - \sum_{j=1}^{\lfloor\frac{q-1}{2}\rfloor}\left(\frac{\chi(j)}{\sin(\frac{\pi j}{q})}\right)^2\right].$$

4. Some multiple L-star series

Consider the “star analog" of $L_n(\chi)$:

$$L_n^*(\chi) := \sum_{1 \le k_1 \le \cdots \le k_n} \frac{\chi(k_1)}{k_1} \cdots \frac{\chi(k_n)}{k_n}.$$

Note: $\le$ instead of $<$ between subscripts.

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Note: $\le$ instead of $<$ between subscripts.

Lemma 9

Let χ be a primitive nonprincipal Dirichlet character. Then

$$\prod_{k=1}^{\infty}\left(1-\chi(k)\frac{z}{k}\right)^{-1} = 1+\sum_{n=1}^{\infty} L_n^*(\chi) z^n.$$

Corollary 10

Let χ be a primitive nonprincipal Dirichlet character. Then for all $n \geq 1$,

$$L_n^*(\chi) + \sum_{j=1}^{n-1} (-1)^j L_j(\chi) L_{n-j}^*(\chi) + (-1)^n L_n(\chi) = 0.$$

Corollary 10

Let χ be a primitive nonprincipal Dirichlet character. Then for all $n \geq 1$,

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For the “easy cases", we need Bernoulli and Euler polynomials:

$$\frac{xe^{zx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!} \qquad (|x| < 2\pi),$$

$$\frac{2e^{zx}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{x^n}{n!} \qquad (|x| < \pi),$$

and Bernoulli and Euler numbers defined by

$$B_n := B_n(0), \qquad E_n := 2^n E_n(\tfrac{1}{2}), \qquad n = 0, 1, 2, \ldots.$$

Corollary 11

(*a*) *If* χ *is the nonprincipal character with* $q = 3$, *then*

$$L_{2n}^*(\chi) = (-1)^{n+1} 3(2^{2n}+1)\frac{B_{2n+1}(\frac{1}{3})}{(2n+1)!}\pi^{2n},$$

$$L_{2n+1}^*(\chi) = (-1)^n \frac{\sqrt{3}}{2}(2^{2n+1}-1)(3^{2n+2}-1)\frac{B_{2n+2}}{(2n+2)!}\left(\frac{\pi}{3}\right)^{2n+1}.$$

(*b*) *If* χ *is the nonprincipal character with* $q = 4$, *then*

$$L_n^*(\chi) = (-1)^{\lfloor \frac{n+1}{2} \rfloor}\frac{E_n(\frac{1}{4})}{n!}\pi^n.$$

(*c*) *If* χ *is the nonprincipal character with* $q = 6$, *then*

$$L_{2n}^*(\chi) = (-1)^n \frac{1}{4}(3^{2n+1}+1)\frac{E_{2n}}{(2n)!}\left(\frac{\pi}{6}\right)^{2n},$$

$$L_{2n+1}^*(\chi) = (-1)^{n+1}\frac{\sqrt{3}}{2}\frac{E_{2n+1}(\frac{1}{6})}{(2n+1)!}\pi^{2n+1}.$$

n	$B_{2n+1}(\frac{1}{3})$	B_{2n+2}	$E_n(\frac{1}{4})$	E_{2n}	$E_{2n+1}(\frac{1}{6})$
0	$-\frac{1}{6}$	$\frac{1}{6}$	1	1	$-\frac{1}{3}$
1	$\frac{1}{27}$	$-\frac{1}{30}$	$-\frac{1}{4}$	-1	$\frac{23}{108}$
2	$-\frac{5}{243}$	$\frac{1}{42}$	$-\frac{3}{16}$	5	$-\frac{1681}{3888}$
3	$\frac{49}{2187}$	$-\frac{1}{30}$	$\frac{11}{64}$	-61	$\frac{257543}{139968}$
4	$-\frac{809}{19683}$	$\frac{5}{66}$	$\frac{57}{256}$	1385	$-\frac{67637281}{5038848}$
5	$\frac{20317}{177147}$	$-\frac{691}{2730}$	$-\frac{361}{1024}$	-50521	$\frac{27138236663}{181398528}$

The general case:

Theorem 12

Let χ be a primitive nonprincipal Dirichlet character with $q > 2$. Then for all $n \geq 1$,

$$L_n^*(\chi)$$
$$= \frac{(-1)^n}{q^n} \sum{}^* \prod_{j \in \Phi} \frac{\chi(j)^{k_j}}{k_j!} \sum_{k=1}^{k_j} B_{k_j,k}\left(\psi(\tfrac{j}{q}), \psi_1(\tfrac{j}{q}), \ldots, \psi_{k_j-k}(\tfrac{j}{q})\right),$$

with index set $\Phi := \{j \mid 1 \leq j \leq q-1, \gcd(j,q) = 1\}$, and summation $\sum^*$ over all $k_j (j \in \Phi)$ that sum to n.

With $n = 2$ and using the relation

$$L_2^*(\chi) + L_2(\chi) = L_1(\chi)L_1^*(\chi) = L_1(\chi)^2 :$$

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Corollary 13

Let χ be a primitive nonprincipal character with $q > 2$. Then

$$\sum_{k=1}^{\infty} \frac{\chi(k)^2}{k^2} = \frac{\pi^2}{q^2} \sum_{j=1}^{\lfloor \frac{q-1}{2} \rfloor} \left(\frac{\chi(j)}{\sin(\frac{\pi j}{q})} \right)^2 .$$

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In particular,

$$\sum_{\substack{k=1 \\ (k,q)=1}}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{q^2} \sum_{\substack{j=1 \\ (j,q)=1}}^{\lfloor \frac{q-1}{2} \rfloor} \csc^2(\tfrac{\pi j}{q}).$$

Example: If q is an odd prime, then the LHS becomes $\zeta(2) - \zeta(2)/q^2$, and with $\zeta(2) = \pi^2/6$, we get

$$\sum_{j=1}^{\frac{q-1}{2}} \csc^2(\tfrac{\pi j}{q}) = \frac{q^2-1}{6}.$$

(This is a well-known identity).

5. Products involving cyclotomic polynomials

Theorem 14

If $m \geq 3$ is an integer that contains a square, then

$$\prod_{k=1}^{\infty} \Phi_m\left(\tfrac{z}{k}\right) = \prod_{\substack{j=1 \\ (j,m)=1}}^{m-1} \frac{1}{\Gamma(1 - ze^{2\pi ij/m})}.$$

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Example 1: Let $m = 4$. Then $\Phi_4(\frac{z}{k}) = 1 + (\frac{z}{k})^2$, and

$$\prod_{k=1}^{\infty} \left(1 + (\tfrac{z}{k})^2\right) = \frac{i}{-\pi z} \sin(\pi iz) = \frac{\sinh(\pi z)}{\pi z},$$

a well-known identity.

Example 2: Let $m = 12$. Then $\Phi_{12}(x) = 1 - x^2 + x^4$, and thus

$$\prod_{k=1}^{\infty}\left(1 - (\tfrac{z}{k})^2 + (\tfrac{z}{k})^4\right) = \frac{-1}{\pi^2 z^2}\sin(\pi z e^{\pi i/6})\sin(\pi z e^{5\pi i/6})$$

$$= \frac{\sin^2(\frac{1}{2}\sqrt{3}\pi z) + \sinh^2(\frac{1}{2}\pi z)}{\pi^2 z^2}.$$

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With $z = 1/(2\sqrt{3})$, we get

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{12k^2} + \frac{1}{144k^4}\right) = \frac{6}{\pi^2}\cdot\cosh\left(\frac{\pi}{2\sqrt{3}}\right).$$

Thank you