Lifting automorphisms of $k[[t]]$ from characteristic p to characteristic 0.

DEF Let R be a ring

$\mathrm{char}(R) = 0$ if $\nexists\, n \in \mathbb{N}$ s.th. $\underbrace{1_R + \dots + 1_R}_{n\text{-times}} = 0$

$\mathrm{char}(R) = \min\{n \in \mathbb{N} \mid n \cdot 1_R = 0\}$ otherwise

If k is a field, then $\mathrm{char}(k)$ is either 0 or a prime number p.

eg $\mathrm{char}(\mathbb{Q}) = 0$, $\mathrm{char}(\mathbb{F}_{p^n}) = p$

REDUCTION: $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$

$$a \longmapsto [a] \bmod p$$

$\{(x,y)\in\mathbb{Z}^2 : x^2+y^2 = 1001\} = \emptyset$ because mod 7

$$x^2+y^2 = [0] \iff (x,y)=(0,0)$$

LIFTING: char p $\longrightarrow$ char 0

Lifting automorphisms of $k[[t]]$

<u>DEF</u> A discrete valuation ring is a principal ideal domain and has a unique non-zero prime ideal $\mathfrak{m} = \langle \pi \rangle$

R DVR $\rightsquigarrow R/_{\mathfrak{m}} = k$ (residue field of R)

<u>eg)</u> $k[[t]] = \{ \sum_{i=0}^{\infty} a_i t^i \,/\, a_i \in k \}$ is a DVR, $\mathfrak{m} = \langle t \rangle$,

$k[[t]]/_{\langle t \rangle} = k$

<u>Theorem</u> (Witt) Let k be a field of characteristic p, perfect

Then there is a unique DVR R minimal for the following

- $R/_{\mathfrak{m}} = k$
- $\mathrm{char}(R) = 0$
- R complete

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e.g.) $k = \mathbb{F}_p \rightsquigarrow R = \mathbb{Z}_p$ (p-adic integers)

$k = \overline{\mathbb{F}}_p \rightsquigarrow R = \widehat{\mathbb{Z}_p^{nr}}$

LOCAL LIFTING PROBLEM

Let k alg. closed field, $\mathrm{char}(k) = p > 0$,
R as above, $G \subset \mathrm{Aut}_k(k[[t]])$ finite subgroup.

$\exists\ G \hookrightarrow \mathrm{Aut}_R(R[[T]])$ s.th.

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr] & \mathrm{Aut}_R(R[[T]]) \arrow[d] \\ & \mathrm{Aut}_k(k[[t]]) \end{tikzcd}$$

commutes ?


Answer no
$G = \mathbb{Z}/m\mathbb{Z} = \langle \sigma \rangle$, $(m,p)=1$
$\sigma(t) = \zeta_m t \rightsquigarrow \sigma(T) = \zeta_m T$
$G = \mathbb{Z}/p\mathbb{Z}$ $\sigma(t) = t+1$
$\sigma(t) = \frac{t}{t+1}$ $\frac{\frac{t}{t+1}}{\frac{t}{t+1}+1} = \frac{t}{2t+1}$ ...
$k((t))$ — $k[[t]]$
G-Galois
$k((z))$ — $k[[t]]^G \cong k[[z]]$
elements fixed by G
$R[[T]] \hookrightarrow \mathrm{Frac}(R[[T]]) - \mathcal{L}$
$R[[Z]] \hookrightarrow \mathrm{Frac}(R[[Z]]) - \mathcal{K}$
Kummer
$\mathcal{L} = \mathcal{K}[W]/(W^p - u)$
if $\zeta_p \in R$
Artin-Schreier ($G = \mathbb{Z}/p\mathbb{Z}$)
$k((t)) = k((z))[y]/(y^p - y - z^{-m})$
$m \in \mathbb{N}$ $(m,p)=1$
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$\underline{m=1}$ $\quad y^p - y = z^{-1} \xrightarrow{\text{lifts to}} ?$

$\lambda = \zeta_p - 1$, $\quad W^p = 1 + \lambda^p z^{-1}$

$v_p(\lambda) = \frac{1}{p-1}$

$$\Big\downarrow \quad W = 1 + \lambda Y$$

$$\frac{(\lambda Y)^p + \dots + p\lambda Y}{\lambda^p} = \frac{\lambda^p z^{-1}}{\lambda^p}$$

$v \geq 1$

$$Y^p + (\dots) + \frac{p}{\lambda^{p-1}} Y = z^{-1} \xrightarrow{\text{reduce}} y^p - y = z^{-1}$$

Thm (Henrio) $y^p - y = z^{-m}$ "lifts to" $W^p - u(Z)$

iff

$\frac{d\bar{u}}{\bar{u}} \in \Omega^1_{\mathbb{P}^1_k}$ has
- $m+1$ distinct poles
- a unique 0 at ∞
- residues in $\mathbb{F}_p^\times$

Def An equidistant lifting is any lifting coming from $\frac{du}{u}$ as above.

Thm (T.) If $G = \left(\mathbb{Z}/3\mathbb{Z}\right)^n$ then there are no equidistant liftings for $m = 3, 9, 15, \dots, \lambda\cdot 3$ for λ odd