

Lifting automorphisms of $k[t]$

DEF Let R be a ring

$$\text{char}(R) = 0 \text{ if } \nexists n \in \mathbb{N} \text{ s.t. } \underbrace{1_R + \dots + 1_R}_{n \text{ times}} = 0$$

$$\text{char}(R) = \min\{n \in \mathbb{N} \mid n \cdot 1_R = 0\} \text{ otherwise}$$

from characteristic p to characteristic 0.

If k is a field, then $\text{char}(k)$ is either 0 or a

prime number p .

eg $\text{char}(\mathbb{Q}) = 0$, $\text{char}(\mathbb{F}_p) = p$

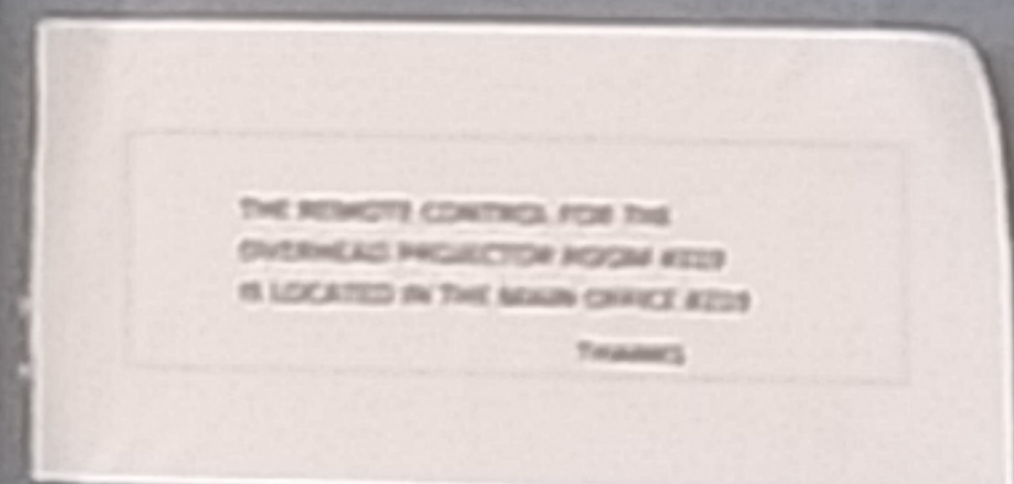
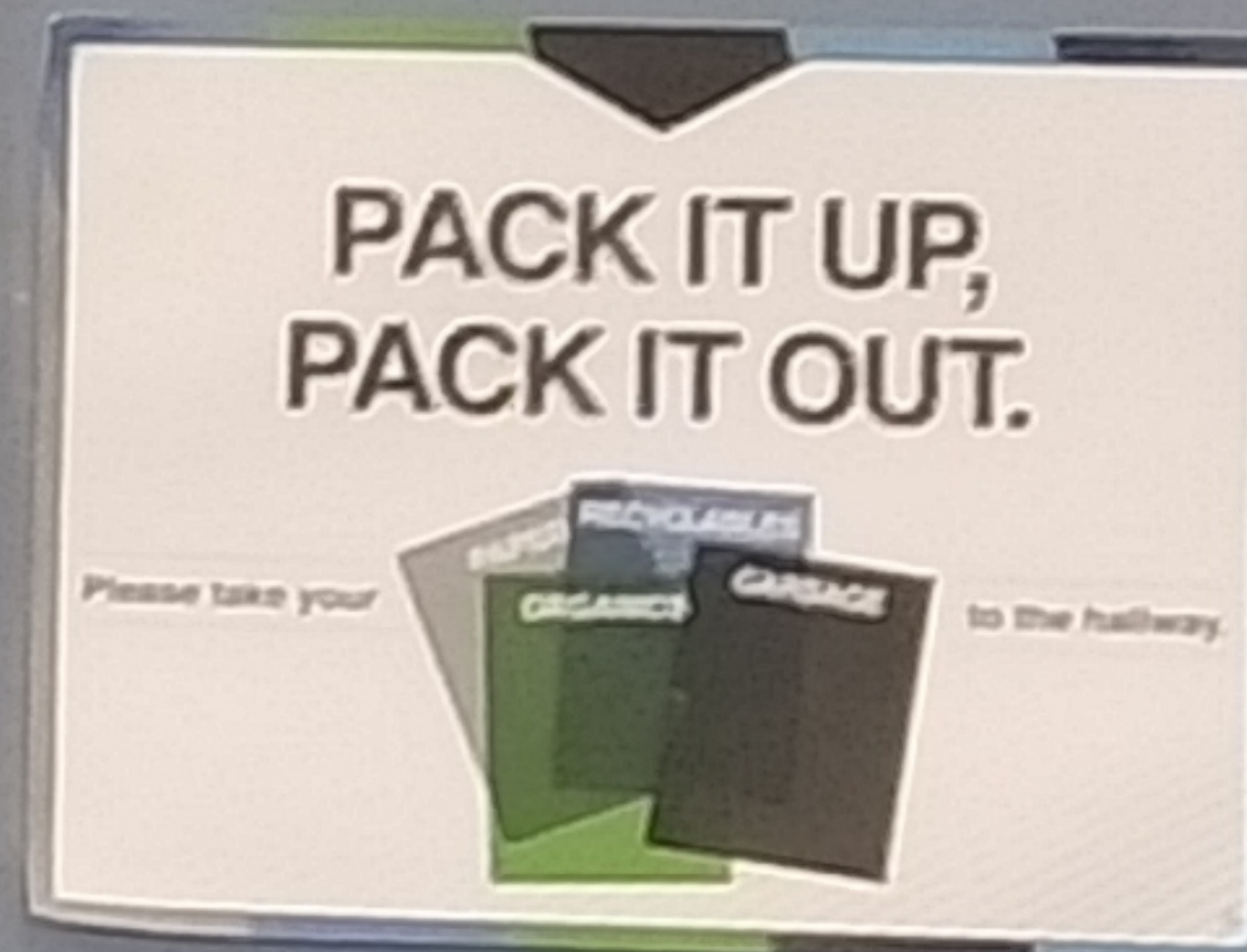


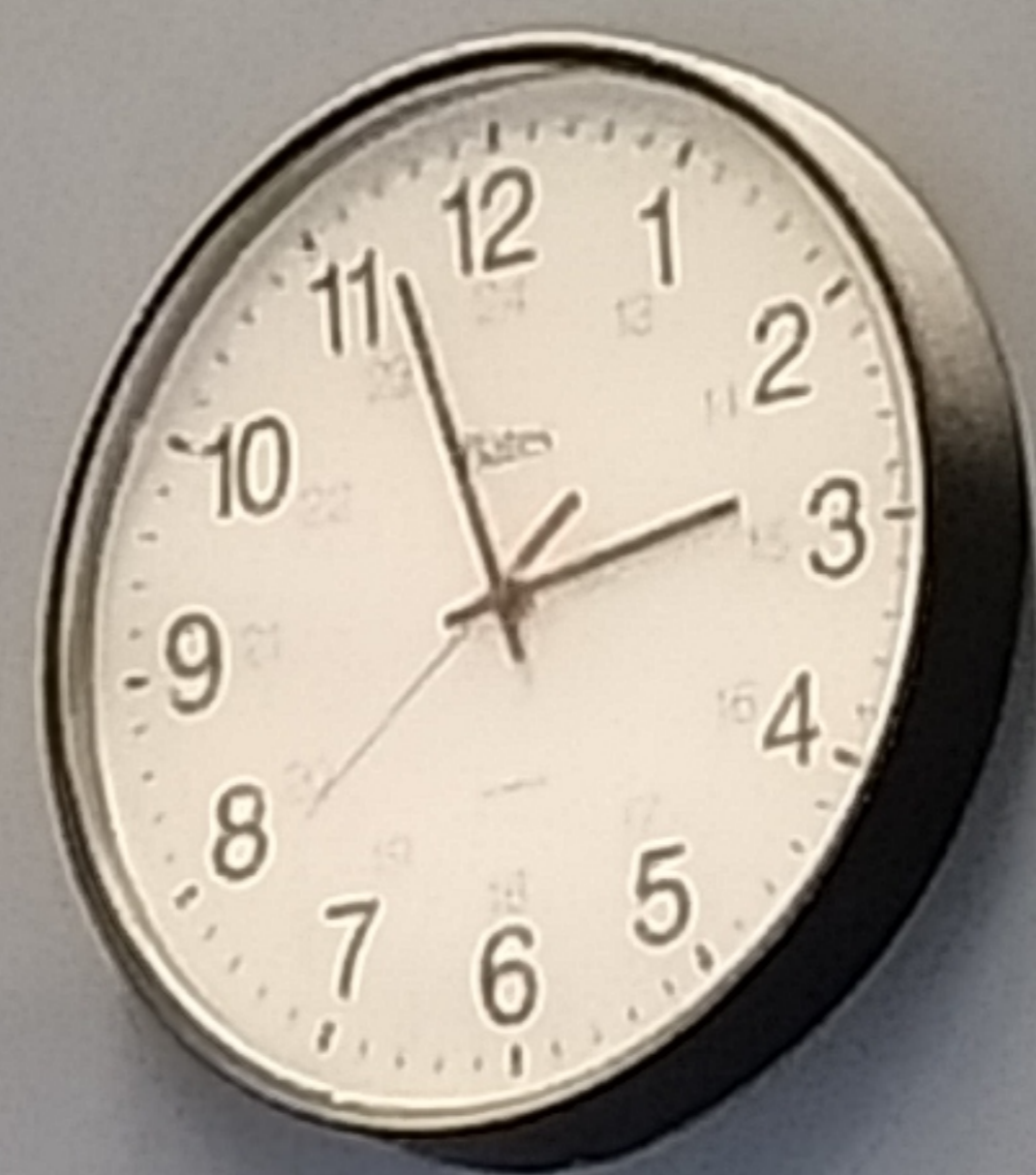
REDUCTION: $\mathbb{Z} \rightarrow \mathbb{F}_p$
 $a \mapsto [a] \text{ mod } p$

$(x, y) \in \mathbb{Z}^2$
 $x^2 + y^2 = 1001 \not\equiv \emptyset \text{ because mod } 7$

$$x^2 + y^2 = [0] \Leftrightarrow (x, y) = (0, 0)$$

LIFTING: $(\text{char } p) \rightarrow (\text{char } 0)$





Lifting automorphisms of $k[[t]]$

DEF A discrete valuation ring is a principal ideal domain and has a unique non-zero prime ideal $\mathfrak{m} = \langle \pi \rangle$

$R_{\text{DVR}} \rightsquigarrow R_{\mathfrak{m}} = k$ (residue field of R)

eg) $k[[t]] = \left\{ \sum_{i=0}^{\infty} a_i t^i \mid a_i \in k \right\}$ is a DVR, $\mathfrak{m} = \langle t \rangle$,

$$k[[t]] / \langle t \rangle = k$$

Theorem (Witt) Let k be a field of characteristic p , perfect

Then there is a unique DVR R minimal for the following

- $R_{\mathfrak{m}} = k$
- $\text{char}(R) = 0$
- R complete



e.g) $k = \mathbb{F}_p \rightsquigarrow R = \mathbb{Z}_p$ (p -adic integers)

$k = \overline{\mathbb{F}_p} \rightsquigarrow R = \widehat{\mathbb{Z}_p}^{\text{nr}}$

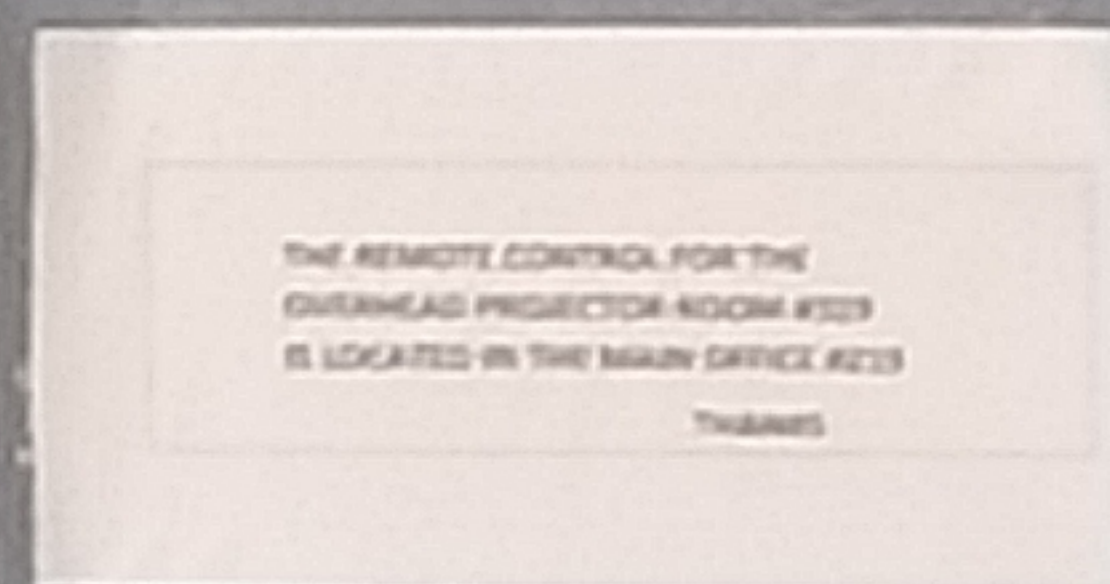
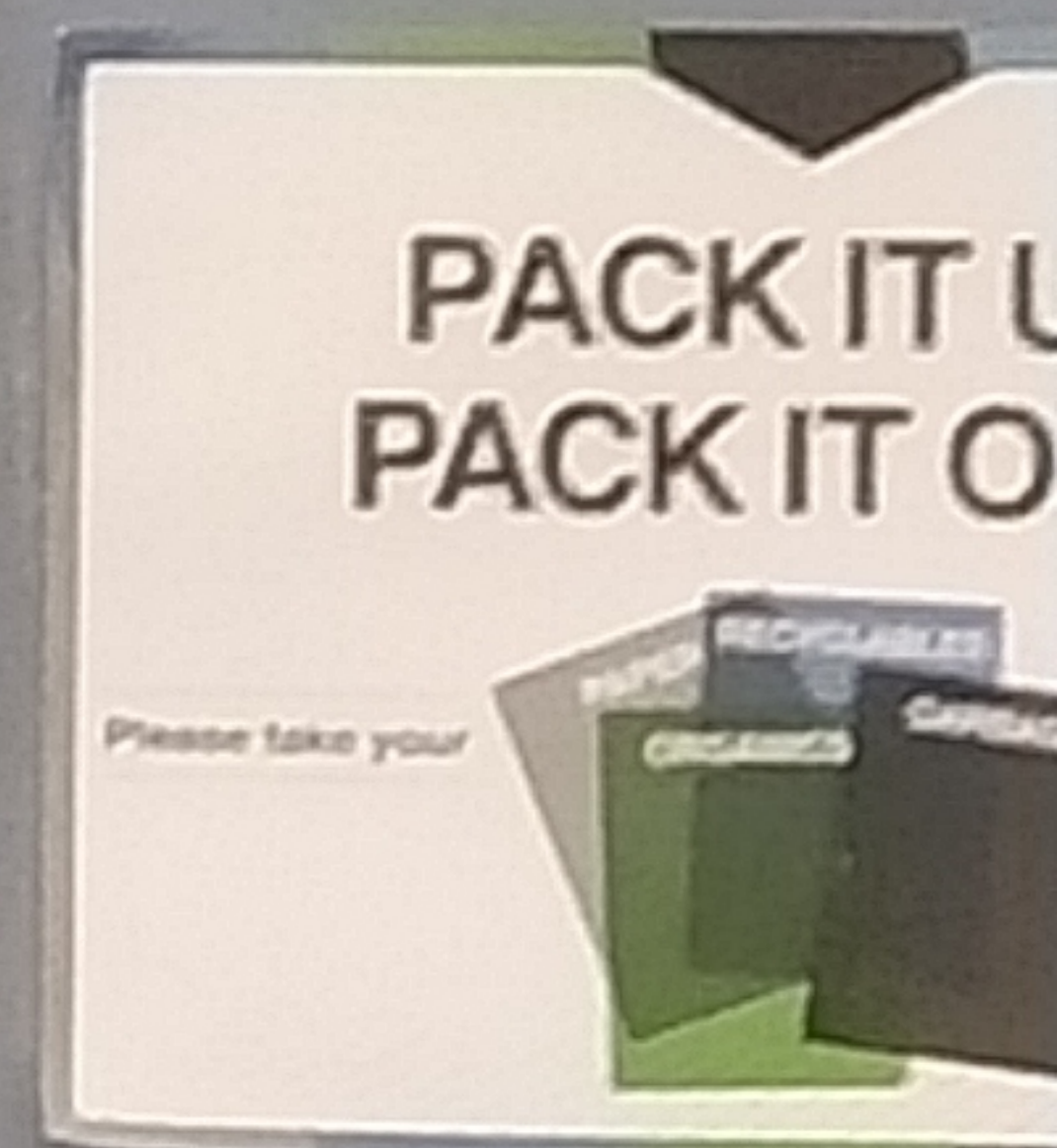
LOCAL LIFTING PROBLEM

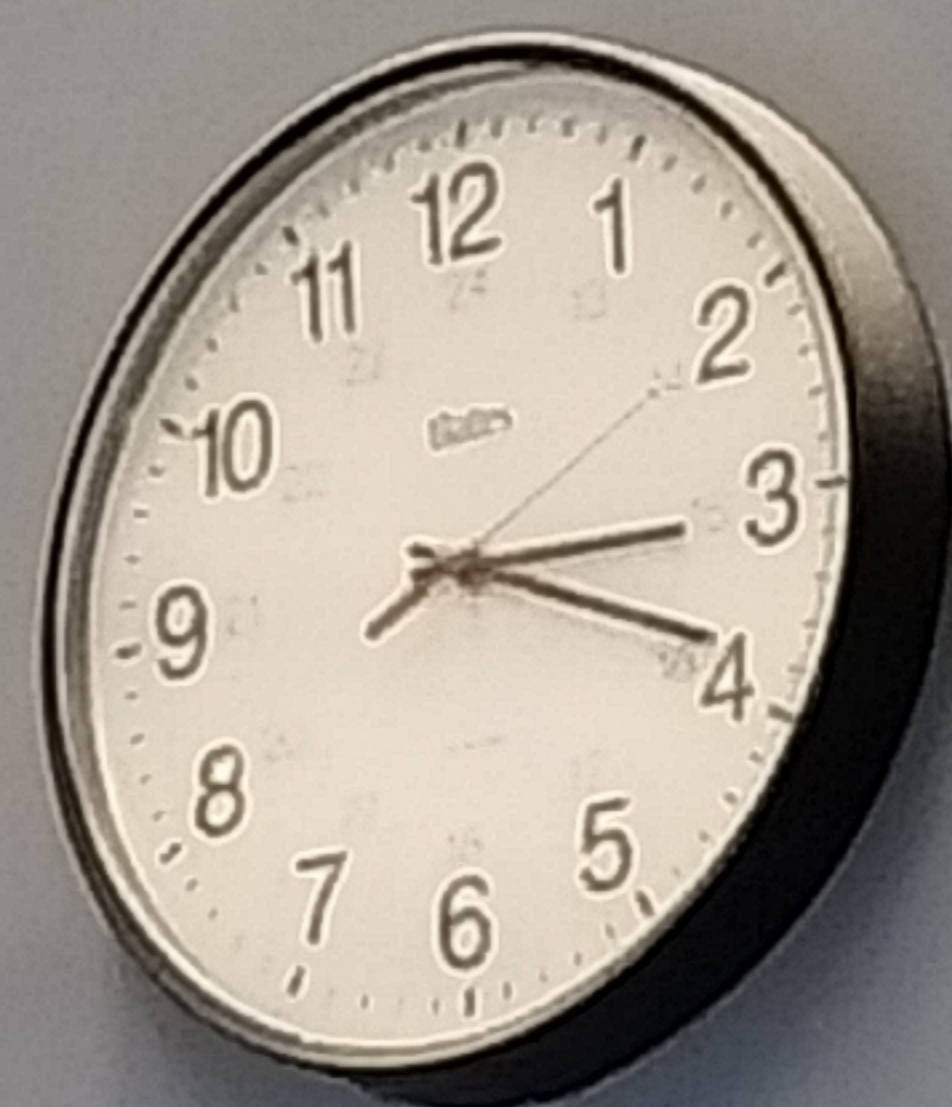
Let k alg closed field, $\text{char}(k) = p > 0$

R as above $G \subset \text{Aut}_k(k[[t]])$ finite subgroup.

$\exists G \hookrightarrow \text{Aut}_k(R[[t]])$ s.t.h.

\searrow
 $\text{Aut}_k(k[[t]]) \downarrow$
 $\text{Aut}_k(R[[t]])$ commutes?





Answer no
 $G = \mathbb{Z}/m\mathbb{Z}$, $(m,p)=1$
 $\langle \sigma \rangle$
 $\sigma(t) = \sum_n t \rightsquigarrow \sigma(T) = \sum_n T$
 $G = \mathbb{Z}/p\mathbb{Z} \cdot \sigma(t) = t+1$
 $\sigma(t) = \frac{t}{t+1}$ $\frac{\frac{t}{t+1}}{\frac{t}{t+1}+1} = \frac{t}{2t+1}$...

$k(t) \xrightarrow{G\text{-tors}} k[t]$
 $k(z) \xrightarrow{G\text{-tors}} k[[t]] \xrightarrow{G} k[[z]]$
 $R[t] \hookrightarrow \text{Frac}(R[[t]]) = \mathbb{L}$
 $R[z] \hookrightarrow \text{Frac}(R[[z]]) = \mathbb{K}$
 Antim-Schreier elements fixed by G
 $(G = \mathbb{Z}/p\mathbb{Z})$
 $k(t) = k(z)[y]$
 $y^p - y = \frac{t}{2t+1}$
 $m \in \mathbb{N}$ $(m,p)=1$
 $\mathbb{L} = \mathbb{K}[\overline{W}]$
 $\overline{W^{p-m}}$
 if $\sum_p \in \mathbb{R}$

$m=1$ $y^p - y = z^{-1}$ \rightsquigarrow lfts to (2)

$\lambda = z_p - 1$, $w^p = 1 + \lambda^p z^{-1}$

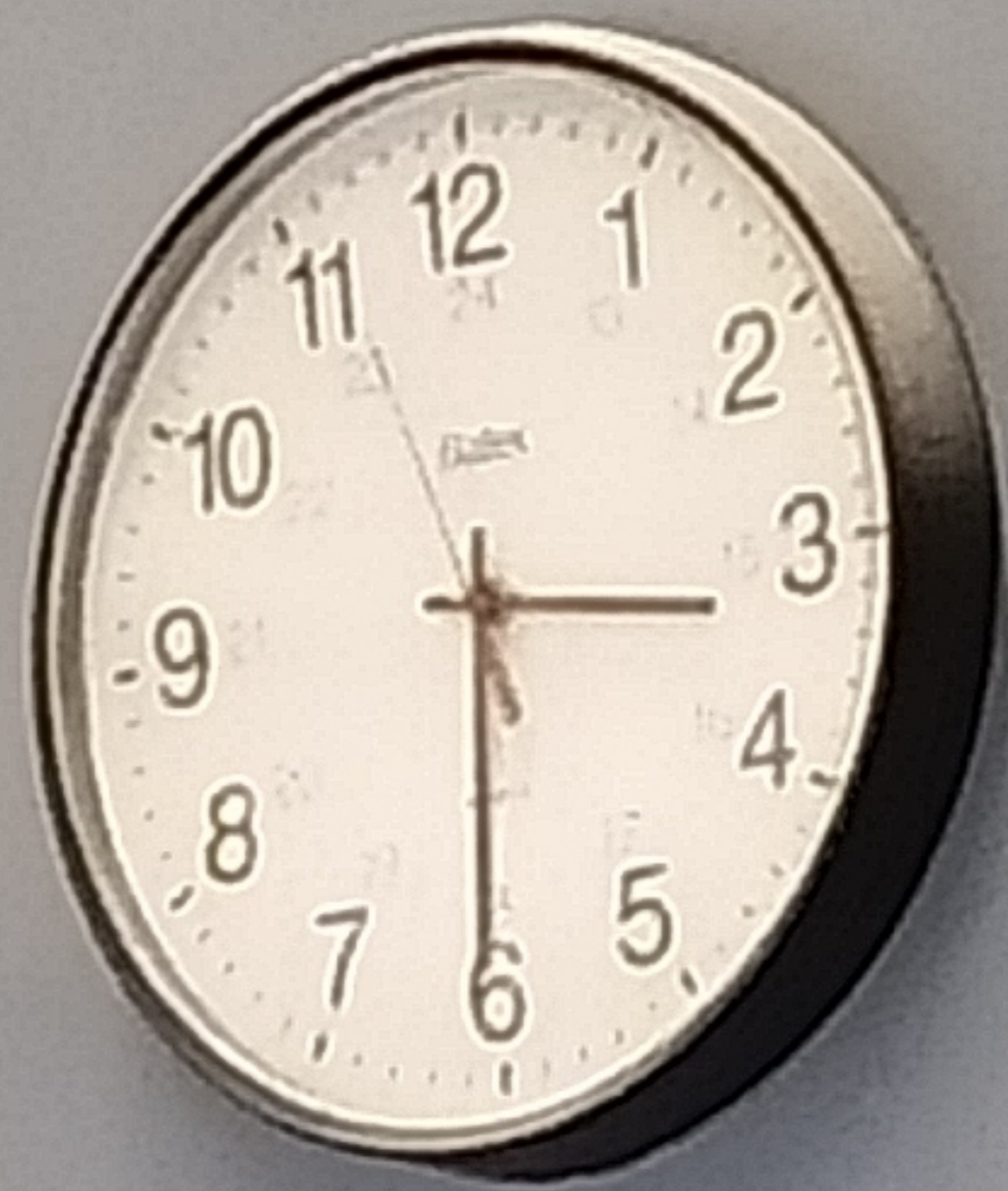
$v_p(\lambda) = \frac{1}{p-1}$

$w = 1 + \lambda z$

$(\lambda z)^p + \dots + p\lambda z = \lambda^p z^{-1}$

$\lambda^p \geq 1$

$y^p + \dots + \frac{p}{\lambda^{p-1}} y = z^{-1}$ \rightsquigarrow reduce $y^p - y = z^{-1}$



Thm (Hensel) $y^p - y = z^{-m}$ "lifts to" $W^p - m(z)$
iff

$\frac{dm}{m} \in \Omega_{\mathbb{F}_k}^1$ has

- $m+1$ distinct poles
- a unique 0 at ∞
- residues in \mathbb{F}_p^\times

Def An equidistant lifting is any lifting coming
from $\frac{dm}{m}$ as above.

Thm (T.) If $G = \left(\frac{z}{3z}\right)^m$ then there are no equidistant
liftings for $m = 3, 9, 15, \dots, \lambda \cdot 3$ for λ odd