Distinct and Complete Integer Partitions

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This is joint work with George Andrews and Brian Hopkins.

Abstract

Two infinite lower-triangular matrices related to integer partitions are inverses of each other. One matrix comes from an analogue of the Möbius mu function, while the other comes from counting generalized complete partitions; a complete partition of n has all possible subsums 1 to n.

Mathematica Definitions

Integer Partitions

Definition

A multiset is a collection of elements (like a set) where an element can occur a finite number of times (unlike a set).

An integer partition λ of a positive integer n is an multiset of positive integers λ_i (called its parts) that sum to n. We write $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m) \vdash n$.

Mathematically we use (round) parentheses and in Mathematica we use (curly) braces, which denotes an (ordered) list, not a set.

For example, $(3, 1, 1) \vdash 5$.

Since the elements of a multiset are unordered (like a set), we can take them to be in nonincreasing order from now on.

Here are the integer partitions of 5:

```
Out[•]= {{5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, {1, 1, 1, 1, 1}}
```

Here they are again more compactly:

```
Out[•]= {5, 41, 32, 311, 221, 2111, 11111}
```

Other Definitions

An older alternative definition is along these lines:

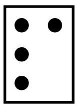
“A partition is a way of writing an integer n as a sum of positive integers where the order of the addends is not significant, …. By convention, partitions are normally written from largest to smallest addends…, for example, 10 = 3 + 2 + 2 + 2 + 1.” (mathworld.wolfram.com/Partition.html)

With such a definition, 3 + 2 + 2 + 2 + 1 has to be frozen, because as an arithmetic expression it is 10 and the parts are gone.

Yet another definition: $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m)$ is an partition of n if the finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_m = n$.

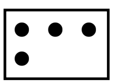
Ferrers Diagram

For each part λ_i of a partition λ, draw a row of λ_i dots, then stack the rows.

Out[•]= 

Conjugate Partition

The conjugate partition λ' of a partition λ is the partition corresponding to the transpose of the Ferrers diagram of λ.

Out[•]= 

So (3, 1) is the conjugate partition of (2, 1, 1) and vice versa.

Distinct Partition

A distinct partition has no repeated part.

Here are the four distinct partitions of 6.

Out[•]= {6, 51, 42, 321}

The remaining partitions of 6 have repeated parts.

Out[•]= {33, 222, 411, 2211, 3111, 21111, 111111}

This is the sequence counting the number of distinct partitions of n.

Out[•]= {1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 18, 22, 27, 32, 38, 46, 54, 64}

Generating Functions

Number of Partitions

The number of partitions of n is 1, 2, 3, 5, 7, 11, … but the next number is not 13:

Out[•]= {1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77}

The generating function for this sequence $p(n)$ is:

Out[ ]= $x + 2 x^2 + 3 x^3 + 5 x^4 + 7 x^5 + 11 x^6 + 15 x^7 + 22 x^8 + 30 x^9 + 42 x^{10} + 56 x^{11} + 77 x^{12} + \ldots$

The generating function is equal to the infinite product $\prod_{i=1}^{\infty} \frac{1}{1-x^i}$:

Number of Distinct Partitions

The number of distinct partitions of *n*:

Out[ ]= {1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15}

The generating function for this sequence $q(n)$ is:

Out[ ]= $x + x^2 + 2 x^3 + 2 x^4 + 3 x^5 + 4 x^6 + 5 x^7 + 6 x^8 + 8 x^9 + 10 x^{10} + 12 x^{11} + 15 x^{12} + \ldots$

It is equal to the infinite product $\prod_{i=1}^{\infty} (1 + x^i)$:

Two Möbius Functions

Square-Free Numbers

A square-free integer is one that is not divisible by a square greater than 1.

Here are the square-free numbers up to 100:

Out[ ]= {1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 69, 70, 71, 73, 74, 77, 78, 79, 82, 83, 85, 86, 87, 89, 91, 93, 94, 95, 97}

Here are numbers up to 100 that are not square-free:

Out[ ]= {4, 8, 9, 12, 16, 18, 20, 24, 25, 27, 28, 32, 36, 40, 44, 45, 48, 49, 50, 52, 54, 56, 60, 63, 64, 68, 72, 75, 76, 80, 81, 84, 88, 90, 92, 96, 98, 99, 100}

Möbius Function μ

In multiplicative number theory the Möbius μ function is defined on the positive integers as follows.

1. If *n* is not square-free, $\mu(n) = 0$.

2. If *n* is square-free, then *n* can be written as the product of *m* distinct primes, for some positive integer *m*. In that case, $\mu(n) = (-1)^m$.

In other words, μ of a square-free integer is −1 or 1 according to whether *n* has an odd or an even number of prime factors.

For example, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$.

Möbius Partition Function μ_P

The function μ_P is the partition analogue of the ordinary Möbius function μ.

Out[•]=

μ	μ_P
product	partition
primes factors	parts
square-free	distinct

Definition of μ_P:

1. Let $\mu_P(\lambda) = 0$ if the partition λ has a repeated part.

2. If the partition λ has distinct parts and m parts in all, $\mu_P(\lambda) = (-1)^m$.

Here are the partitions of 6 and the corresponding values of the Möbius partition function μ_P:

Out[•]=

6	-1
51	1
42	1
411	0
33	0
321	-1
3111	0
222	0
2211	0
21111	0
111111	0

Infinite Triangular Matrices

Pascal’s Triangle

The prime example of an infinite lower-triangular matrix is Pascal’s triangle T. Imagine that the rows keep going down and the columns keep going to the right.

For readability, replace zeros with dots.

Out[•]//MatrixForm=

1	.	.	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.	.	.
1	2	1	.	.	.	.	.	.	.
1	3	3	1	.	.	.	.	.	.
1	4	6	4	1	.	.	.	.	.
1	5	10	10	5	1	.	.	.	.
1	6	15	20	15	6	1	.	.	.
1	7	21	35	35	21	7	1	.	.
1	8	28	56	70	56	28	8	1	.
1	9	36	84	126	126	84	36	9	1

Here is the matrix product $T \cdot T$.

Out[∘]//MatrixForm=

1	.	.	.	.	.	.	.	.	.
2	1	.	.	.	.	.	.	.	.
4	4	1	.	.	.	.	.	.	.
8	12	6	1	.	.	.	.	.	.
16	32	24	8	1	.	.	.	.	.
32	80	80	40	10	1	.	.	.	.
64	192	240	160	60	12	1	.	.	.
128	448	672	560	280	84	14	1	.	.
256	1024	1792	1792	1120	448	112	16	1	.
512	2304	4608	5376	4032	2016	672	144	18	1

Here is the matrix inverse of *T*.

Out[∘]//MatrixForm=

1	.	.	.	.	.	.	.	.	.
-1	1	.	.	.	.	.	.	.	.
1	-2	1	.	.	.	.	.	.	.
-1	3	-3	1	.	.	.	.	.	.
1	-4	6	-4	1	.	.	.	.	.
-1	5	-10	10	-5	1	.	.	.	.
1	-6	15	-20	15	-6	1	.	.	.
-1	7	-21	35	-35	21	-7	1	.	.
1	-8	28	-56	70	-56	28	-8	1	.
-1	9	-36	84	-126	126	-84	36	-9	1

Stirling Numbers of the First and Second Kind

The Stirling numbers of the first and second kind are another example of a pair of inverse lower-triangular matrices.

A Stirling number of the first kind counts how many permutations of {1, 2, …, *n*} have *k* cycles.

Out[∘]//MatrixForm=

1	.	.	.	.	.	.	.
-1	1	.	.	.	.	.	.
2	-3	1	.	.	.	.	.
-6	11	-6	1	.	.	.	.
24	-50	35	-10	1	.	.	.
-120	274	-225	85	-15	1	.	.
720	-1764	1624	-735	175	-21	1	.
-5040	13068	-13132	6769	-1960	322	-28	1

A set partition of a finite set, say *T* = {1, 2, 3, …, *n*}, is a set of disjoint nonempty subsets of *T*.

A Stirling number of the second kind counts how many set partitions of {1, 2, …, *n*} have *k* subsets.

Out[∘]//MatrixForm=

1	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.
1	3	1	.	.	.	.	.
1	7	6	1	.	.	.	.
1	15	25	10	1	.	.	.
1	31	90	65	15	1	.	.
1	63	301	350	140	21	1	.
1	127	966	1701	1050	266	28	1

The two matrices are inverses of each other.

Out[ ]=

$$
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & -3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
-6 & 11 & -6 & 1 & \cdot & \cdot & \cdot & \cdot \\
24 & -50 & 35 & -10 & 1 & \cdot & \cdot & \cdot \\
-120 & 274 & -225 & 85 & -15 & 1 & \cdot & \cdot \\
720 & -1764 & 1624 & -735 & 175 & -21 & 1 & \cdot \\
-5040 & 13068 & -13132 & 6769 & -1960 & 322 & -28 & 1
\end{pmatrix} \cdot
$$

$$
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 7 & 6 & 1 & \cdot & \cdot & \cdot & \cdot \\
1 & 15 & 25 & 10 & 1 & \cdot & \cdot & \cdot \\
1 & 31 & 90 & 65 & 15 & 1 & \cdot & \cdot \\
1 & 63 & 301 & 350 & 140 & 21 & 1 & \cdot \\
1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1
\end{pmatrix} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

Infinite Matrices Can Be Weird

For square matrices A and B, if $A \cdot B = I$, then $B \cdot A = I$.

As the Demonstration The Derivative and the Integral as Infinite Matrices shows, there are (very familiar) infinite matrices $\mathcal{D}$ and $\mathcal{I}$ such that $\mathcal{D} \cdot \mathcal{I}$ is the identity matrix, but $\mathcal{D} \cdot \mathcal{I} \neq \mathcal{I} \cdot \mathcal{D}$.

Even though infinite lower-triangular matrices with 1's on the main diagonal behave well, we only deal with $r \times r$ matrices, where $r \in \mathbb{Z}^+$.

Matrix v

Define the $r \times r$ matrix v_r by $v_r(n, p) = -\sum \mu_P(\lambda)$, where the sum is over $\lambda \vdash n$ and $\max(\lambda) = p$, $1 \le n, p \le r$.

An equivalent definition is that the n, p entry is:

(the number of distinct partitions of n with an odd number of parts)

−

(the number of distinct partitions of n with an even number of parts),
all with maximum part p.

Here is v_{10}:

Out[ ]//MatrixForm=

$$
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & -1 & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & -1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & -1 & -1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & -1 & -1 & 1 & \cdot \\
\cdot & \cdot & \cdot & -1 & 2 & \cdot & \cdot & -1 & -1 & 1
\end{pmatrix}
$$

To verify that $v_{10}(10, 5) = 2$, look at the partitions of 10:

Out[•]= {10, 91, 82, 811, 73, 721, 7111, 64, 631, 622, 6211, 61111, 55, 541, 532, 5311, 5221, 52111, 511111, 442, 4411, 433, 4321, 43111, 4222, 42211, 421111, 4111111, 3331, 3322, 33211, 331111, 32221, 322111, 3211111, 31111111, 22222, 222211, 2221111, 22111111, 211111111, 1111111111}

The ones with maximum part 5 are:

Out[•]= {{5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1, 1}}

Applying μ_P to each of those gives:

Out[•]= {0, -1, -1, 0, 0, 0, 0}

Minus the sum is 2, so $v_{10}(10, 5) = 2$, as claimed.

Inverse of *v*

Jacobi wrote, “Always invert!” (referring to elliptic integrals).

This is v_{15}^{-1}.

Out[•]//MatrixForm=

1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
.	1	1	1	.	.	.	.	.	.	.	.	.	.	.
.	2	2	1	1	.	.	.	.	.	.	.	.	.	.
.	2	2	2	1	1	.	.	.	.	.	.	.	.	.
.	4	4	3	2	1	1	.	.	.	.	.	.	.	.
.	5	5	4	3	2	1	1	.	.	.	.	.	.	.
.	8	8	6	5	3	2	1	1	.	.	.	.	.	.
.	10	10	9	6	5	3	2	1	1	.	.	.	.	.
.	16	16	13	10	7	5	3	2	1	1	.	.	.	.
.	20	20	17	13	10	7	5	3	2	1	1	.	.	.
.	31	31	25	20	14	11	7	5	3	2	1	1	.	.
.	39	39	33	26	20	14	11	7	5	3	2	1	1	.
.	55	55	46	37	28	21	15	11	7	5	3	2	1	1

What is the sequence in the second column, 1, 1, 1, 2, 2, 4, 5, 8, 10, 16, 20, …?

Look it up at the OEIS to find A126796 Number of complete partitions of n.

Matrix *γ*

Subpartitions and Subsums of a Partition

A subpartition of a partition λ is a submultiset of λ. For instance, (3, 1) is a subpartition of (3, 1, 1).

A subsum is the sum of a subpartition. So there are eight ($8 = 2^3$) subsums of (3, 1, 1) corresponding to the eight subpartitions of (3, 1, 1}:

Out[•]=

subpartition	subsum
{}	0
{3}	3
{1}	1
{1}	1
{3, 1}	4
{3, 1}	4
{1, 1}	2
{3, 1, 1}	5

Complete Partition

Define a partition $\lambda \vdash n$ to be complete if it has all possible subsums 1, 2, 3, …, n.

Here are the five complete partitions of 6.

Out[•]= {321, 3111, 2211, 21111, 111111}

Here are the partitions of 6 that are not complete.

Out[•]= {6, 51, 42, 411, 33, 222}

This is the sequence counting the number $c(n)$ of complete partitions of n.

Out[•]= {1, 1, 1, 2, 2, 4, 5, 8, 10, 16, 20}

Park's Condition

Clearly, a complete partition needs to have 1 as a part.

To get the subsum 2, either 2 is a part or {1, 1} is a subpartition.

To get the subsum 3, 3 is a part or either of the subpartitions {1, 1, 1} or {2, 1}.

If a part is too large relative to the others, intermediate subsums fail to appear.

Theorem (Park): A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ is complete iff $\lambda_m = 1$ and for each j, $1 \leq j < m$, $\lambda_j \leq 1 + \lambda_{j+1} + \lambda_{j+2} + \ldots + \lambda_m$.

For example, 411 is not complete (no subsum is 3) because $4 \not\leq 1 + (1 + 1)$.

Exercise: The conjugate of a distinct partition is a complete partition.

k-Step Partition

Given a nonnegative integer k, define a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ to be k-step iff $\lambda_m \leq k$ and for each j, $0 \leq j \leq m$, $\lambda_j \leq k + \lambda_{j+1} + \lambda_{j+2} + \ldots + \lambda_m$.

Define the empty partition to be the only 0-step partition.

Clearly, a 1-step partition is a complete partition.

Out[•]= {311, 221, 2111, 11111}

Here are the k-step partitions of 5, for $k = 1, 2, 3, 4, 5$.

Out[•]= {311, 221, 2111, 11111}

Out[•]= {32, 311, 221, 2111, 11111}

Out[•]= {41, 32, 311, 221, 2111, 11111}

Out[•]= {41, 32, 311, 221, 2111, 11111}

This is the same as the partitions of 5 with no restrictions.

Out[•]= {5, 41, 32, 311, 221, 2111, 11111}

Matrix of Number of *k*-step Partitions

Define $l(n, k)$ to be the number of *k*-step partitions of *n*.

Out[•]//MatrixForm=

1	1	1	1	1	1	1	1	1	1
·	1	1	1	1	1	1	1	1	1
·	1	2	2	2	2	2	2	2	2
·	2	2	3	3	3	3	3	3	3
·	2	4	4	5	5	5	5	5	5
·	4	5	6	6	7	7	7	7	7
·	5	8	9	10	10	11	11	11	11
·	8	10	13	13	14	14	15	15	15
·	10	16	17	20	20	21	21	22	22
·	16	20	25	26	28	28	29	29	30

The second column is the number of complete partitions of *n* is $c(n) = l(n, 2)$.

Definition of γ

Define the matrix γ_r by $\gamma(i, j) = l(i - j, j - 1), i \leq i, j \leq r$.

That is, the columns of γ are the number of *k*-step partitions shifted down to form a lower-triangular matrix.

Here is the matrix γ_{10}:

Out[•]//MatrixForm=

1	·	·	·	·	·	·	·	·	·
·	1	·	·	·	·	·	·	·	·
·	1	1	·	·	·	·	·	·	·
·	1	1	1	·	·	·	·	·	·
·	2	2	1	1	·	·	·	·	·
·	2	2	2	1	1	·	·	·	·
·	4	4	3	2	1	1	·	·	·
·	5	5	4	3	2	1	1	·	·
·	8	8	6	5	3	2	1	1	·
·	10	10	9	6	5	3	2	1	1

It matches the inverse of v_{10}:

Out[]//MatrixForm=

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 2 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 2 & 2 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & 4 & 3 & 2 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 5 & 5 & 4 & 3 & 2 & 1 & 1 & \cdot & \cdot \\ \cdot & 8 & 8 & 6 & 5 & 3 & 2 & 1 & 1 & \cdot \\ \cdot & 10 & 10 & 9 & 6 & 5 & 3 & 2 & 1 & 1 \end{pmatrix}$$

$v^{-1} = \gamma$

Theorem. For each $r \geq 1$, $v_r \cdot \gamma_r = I_r$, the identity matrix.

Up to $r = 6$,

Out[]=

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & -1 & 1 & \cdot \\ \cdot & \cdot & 1 & -1 & -1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & 2 & 2 & 1 & 1 & \cdot \\ \cdot & 2 & 2 & 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Hanna’s Generating Function

Hanna conjectured that

$$1 = \sum_{n=0}^{\infty} c(n)\, q^n (1-q)(1-q^2)\ldots(1-q^{n+1}), \qquad (1)$$

where $c(n)$ is the sequence that counts the number of complete partitions of n.

Proof

Rewrite the desired identity as

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} \frac{c(n)\, q^n}{(1-q^{n+1})(1-q^{n+2})(1-q^{n+3})\ldots} \qquad (2)$$

or

$$\sum_{n=1}^{\infty} p(n)\, q^n = \sum_{n=0}^{\infty} \frac{\sum q^{\pi}}{(1-q^{n+1})(1-q^{n+2})(1-q^{n+3})\ldots}, \qquad (3)$$

where the last sum is over all complete partitions π of n.

Claim: Every partition contains a maximal complete subpartition. For example, (9, 7, 3, 1, 1) has maximal complete subpartition (3, 1, 1). If the maximal subpartition π' of π partitions n, then $n + 1$ cannot be a part of the original partition π. If it were, we could insert it into π', contradicting its maximality.

Furthermore, there is no constraint on the parts in π larger than $n + 1$ because the fact that $n + 1$ is missing in π means that no larger complete subpartition can be produced.

Hence $\frac{\sum q^{\pi}}{\prod_{j=n+2}^{\infty}(1-q^j)} = \frac{c(n)\, q^n}{\prod_{j=n+2}^{\infty}(1-q^j)}$ generates all partitions whose maximal complete subpartition is a partition of n.

Summing over all $n \geq 0$ gives (3) and consequently (1). ■

Identifying coefficients for like powers of q proves that $v \cdot c = (1, 0, 0, 0, \ldots)$, the second column of γ. The straightforward bookkeeping generalization $1 = \sum_{n=0}^{\infty} l(n, k)\, q^n (1-q)(1-q^2)\ldots(1-q^{n+k})$ then proves the theorem for the other columns.

Combinatorial Proof

Here is a proof by example.

Consider the dot product of row 10 of v with c.

Out[•]= $$\begin{pmatrix} 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 4 \\ 5 \\ 8 \\ 10 \end{pmatrix}$$

An entry from v is the difference between the number of distinct partitions of odd and even length. Here are these partitions.

Out[•]= {{}, {}, {}, {4321}, {541, 532}, {64, 631}, {73, 721}, {82}, {91}, {10}}

Here are the complete partitions counted in the third column of γ.

Out[•]//MatrixForm=

```
{}
{}
{1}
{11}
{21, 111}
{211, 1111}
{311, 221, 2111, 11111}
{321, 3111, 2211, 21111, 111111}
```

(Recall the number of complete partitions sequence starts like this:)

Out[•]= {1, 1, 2, 2, 4, 5}

Consider the fifth term in the dot product: 2×2. It comes from all possible pairs {541, 532} × {{21, 111}}.

That is,

{541, 21},

{541, 111},

{532, 21},

{532, 111}.

We will find four other terms in the dot product of opposite sign to get cancellation.

Involution β

Let $\mathcal{D}$ be the set of distinct partitions and $\mathcal{C}$ be the set of complete partitions.

Define $\beta : \mathcal{D} \to \mathcal{C}$ as follows.

Let $d = (d_1, d_2, d_3, \ldots, d_m) \in \mathcal{D}$ and $c = (c_1, c_2, c_3, \ldots) \in \mathcal{C}$.

1. If m is even, then $\beta(d, c) = (d_1 + d_2, d_3, \ldots, d_m), (d_2, c_1, c_2, c_3, \ldots))$.

2. If m is odd, then $\beta(d, c) = ((d_1 - c_1, c_1, d_2, d_3, \ldots, d_m), (c_2, c_3, \ldots))$.

In words:

1. Add the second-largest part d_2 of d to the first part d_1 and adjoin d_2 to c.

2. Drop the largest part c_1 of c from c and in d, subtract c_1 from the largest part d_1 and adjoin c_1 to d.

In the example,

$\beta(541, 21) = (91, 421)$,
$\beta(541, 111) = (91, 4111)$,
$\beta(532, 21) = (82, 321)$,
$\beta(532, 111) = (82, 3111)$.

The resulting pairs are still (distinct, complete).

The function β changes the parity of the length of the distinct partition and is an involution on the set of pairs. Therefore the dot product is zero. "■"

Compositions

A composition of n is a finite sequence of nonnegative integers with sum n. So unlike an integer partition, order matters. For example the two compositions (1, 0, 2) and (1, 2, 0) are different.

Allowing 0 as a part only make sense if the number of parts is specified.

Strict Compositions

A strict composition of n is a finite sequence of positive integers with sum n.

Here are the strict compositions of 4.

```
Out[•]= {4, 31, 13, 22, 211, 121, 112, 1111}
```

Let $L(s)$ be the number of parts of the composition s. Here are the lengths of the compositions just shown:

```
Out[•]= {1, 2, 2, 2, 3, 3, 3, 4}
```

Matrix σ

Like ν is for partitions, so is σ for strict compositions.

Define the $r \times r$ matrix σ_r by $\sigma(n, m) = -\sum(-1)^{\#(s)}$, where $1 \le n \le r$, $1 \le m \le r$. The sum is over all strict compositions c of n with maximum part m and $\#(s)$ is the number of parts of s.

For example, for $n = 4$, $m = 2$, these are the strict compositions:

```
Out[•]= {22, 211, 121, 112}
```

Three have odd length and one has even length, so $\sigma(4, 2) = 3 - 1 = 2$. (Every math talk has some arithmetic.)

Define the $r \times r$ matrix σ_r by $\sigma(n, m)$, $1 \leq n, m \leq r$.

Out[∘]=

1	·	·	·	·	·	·	·	·	·
−1	1	·	·	·	·	·	·	·	·
1	−2	1	·	·	·	·	·	·	·
−1	2	−2	1	·	·	·	·	·	·
1	−1	1	−2	1	·	·	·	·	·
−1	·	1	1	−2	1	·	·	·	·
1	·	−1	·	1	−2	1	·	·	·
−1	1	−1	1	·	1	−2	1	·	·
1	−2	2	−1	·	·	1	−2	1	·
−1	2	−1	−1	1	·	·	1	−2	1

Inverse of σ

Take the inverse of σ_{10}. What are these numbers?

Out[∘]=

1	·	·	·	·	·	·	·	·	·
1	1	·	·	·	·	·	·	·	·
1	2	1	·	·	·	·	·	·	·
1	2	2	1	·	·	·	·	·	·
1	3	3	2	1	·	·	·	·	·
1	2	3	3	2	1	·	·	·	·
1	3	4	4	3	2	1	·	·	·
1	3	4	4	4	3	2	1	·	·
1	3	4	5	5	4	3	2	1	·
1	2	4	5	5	5	4	3	2	1

To answer, define two lower-triangular matrices α and χ.

Matrix α

Let α be the lower-triangular matrix of all 1's:

Out[∘]=

1	·	·	·	·	·	·	·	·	·
1	1	·	·	·	·	·	·	·	·
1	1	1	·	·	·	·	·	·	·
1	1	1	1	·	·	·	·	·	·
1	1	1	1	1	·	·	·	·	·
1	1	1	1	1	1	·	·	·	·
1	1	1	1	1	1	1	·	·	·
1	1	1	1	1	1	1	1	·	·
1	1	1	1	1	1	1	1	1	·
1	1	1	1	1	1	1	1	1	1

Matrix χ

Define the lower-triangular matrix χ by $\chi(n, k) = \begin{cases} \mu\left(\frac{n}{k}\right) & \text{if } k \mid n \\ 0 & \text{otherwise} \end{cases}$

where $1 \leq k \leq n$.

Out[•]=

1	.	.	.	.	.	.	.	.	.
-1	1	.	.	.	.	.	.	.	.
-1	.	1	.	.	.	.	.	.	.
.	-1	.	1	.	.	.	.	.	.
-1	.	.	.	1	.	.	.	.	.
1	-1	-1	.	.	1	.	.	.	.
-1	.	.	.	.	.	1	.	.	.
.	.	.	-1	.	.	.	1	.	.
.	.	-1	.	.	.	.	.	1	.
1	-1	.	.	-1	.	.	.	.	1

Conjecture

$\sigma^{-1} = \alpha \cdot \chi \cdot \alpha$.

The relevant OEIS triangles are A134542, A134541, A000012, A054525.

References

(George Andrews, George Beck, Brian Hopkins) On a Conjecture of Hanna Connecting Distinct Part and Complete Partitions (accepted, Annals of Combinatorics)

The notation there is a little different.

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