Definitions for “Distinct and Complete Integer Partitions”

Multiset

A multiset is a collection of elements (like a set) which can occur with multiplicity (unlike a set).

Integer Partition

An integer partition λ of a positive integer n is a multiset of positive integers λ_i (called its parts) that sum to n. We write $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m) \vdash n$.

Number of Partitions $p(n)$

The function $p(n)$ is the number of partitions of n.

Distinct Partition

A distinct partition has no repeated part.

Number of Distinct Partitions

The function $q(n)$ is the number of distinct partitions of n.

Möbius Function μ

If there is an $r \in \mathbb{Z}^+$ such that $r^2 \mid n$, then $\mu(n) = 0$.

Otherwise n can be written as the product of m distinct primes, for some $m \in \mathbb{Z}^+$; then $\mu(n) = (-1)^m$.

Möbius Partition Function μ_P

Definition of μ_P:

If the partition λ has a repeated part, $\mu_P(\lambda) = 0$.

If the partition λ has distinct parts and m parts in all, $\mu_P(\lambda) = (-1)^m$.

Matrix v

Define the $r \times r$ matrix v_r by $v_r(n, p) = -\sum \mu_P(\lambda)$, where the sum is over all partitions λ of n with $\max(\lambda) = p$, where $1 \leq n \leq r$, $1 \leq p \leq r$.

Subpartition of a Partition

A subpartition of a partition λ is a submultiset of λ.

Subsum of a Partition

A subsum is the sum of a subpartition.

Complete Partition

A partition $\lambda \vdash n$ is complete if its subpartitions have all possible subsums 1, 2, 3, …, n.

k-Step Partition

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ to be k-step iff $\lambda_m \leq k$ and for each j, $0 \leq j \leq m$, we have the inequality $\lambda_j \leq k + \lambda_{j+1} + \lambda_{j+2} + \ldots + \lambda_m$.

From Park's condition, a 1-step partition is complete.

Matrix of Number of k-step Partitions

Define $l(n, k)$ to be the number of k-step partitions of n.

Matrix γ

Define the matrix γ_r by $\gamma(i, j) = l(i - j, j - 1)$, where $i \leq i \leq r$, $i \leq j \leq r$.

That is, the columns of γ are the number of k-step partitions shifted down to form a lower-triangular matrix.

Involution β

Let $\mathcal{D}$ be the set of distinct partitions and C be the set of complete partitions.

Define $\beta : \mathcal{D} \to C$ as follows.

Let $d = (d_1, d_2, d_3, \ldots, d_m) \in \mathcal{D}$ and $c = (c_1, c_2, c_3, \ldots) \in C$.

1. If m is even, then $\beta(d, c) = (d_1 + d_2, d_3, \ldots, d_m), (d_2, c_1, c_2, c_3, \ldots))$.

2. If m is odd, then $\beta(d, c) = ((d_1 - c_1, c_1, d_2, d_3, \ldots, d_m), (c_2, c_3, \ldots))$.

Strict Composition

A strict composition of n is a finite sequence of positive integers with sum n.

Matrix σ

Define the $r \times r$ matrix σ_r by $\sigma(n, m) = -\sum(-1)^{\sharp(s)}$, where $1 \leq n \leq r$, $1 \leq m \leq r$. The sum is over all strict compositions c of n with maximum part m and $\sharp(s)$ is the number of parts of s.

Matrix α

Let α be the lower-triangular matrix of all 1's.

Matrix χ

Define the lower-triangular $n \times n$ matrix χ by $\chi(n, k) = \begin{cases} \mu\left(\frac{n}{k}\right) & \text{if } k \mid n \\ 0 & \text{otherwise} \end{cases}$

where $1 \le k \le n$.