BREAKING DOWN HOMOLOGICAL CYCLES OF SIMPLICIAL COMPLEXES AND THE SUBADDITIVITY PROPERTY OF SYZYGIES

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Ideals in Commutative Algebra

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$$y = -x \qquad y = \pm x$$

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Commutative algebra studies ideals generated by polynomials.

Monomial ideals

A special class of ideals are those generated by **monomials**, which are products of variables.

Example. The ideal (x^2, xyz^3, y^{10}).

For monomial ideals, there are many combinatorial methods for studying their properties.

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An ideal, like a vector space, has a generating set, but most often there is no basis (no linear independence!).

If $f = x^2$, $g = xyz^3$, $h = y^{10}$ then $yz^3 f - xg = 0$.

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Using syzygies, we represent an ideal by a sequence of vector spaces:

$$\cdots \to \mathbb{R}^{a_n} \to \mathbb{R}^{a_{n-1}} \to \cdots \to \mathbb{R}^{a_1} \to \mathbb{R}^{a_0} = 3$$

We can do this so that the a_i are the smallest possible integers, and we call this a **minimal free resolution** of the ideal, which is unique (up to isomorphism).

The minimal free resolution

If a monomial ideal I has minimal graded free resolution

$$\cdots \to \mathbb{R}^{\beta_n} \to \mathbb{R}^{\beta_{n-1}} \to \cdots \to \mathbb{R}^{\beta_1} \to \mathbb{R}^{\beta_0}$$

The β_i are called the **betti numbers** of I.

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[Gasharov-Peeva-Welker] The betti numbers of I can be extracted from the **lcm lattice** of I.

The lcm lattice

Example. $I = (ab, bc, cd)$

ab bc cd

The lcm lattice

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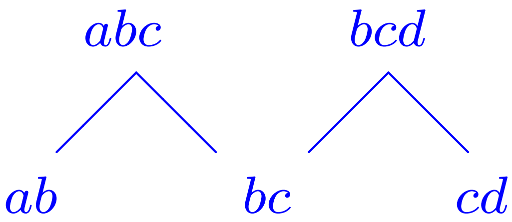

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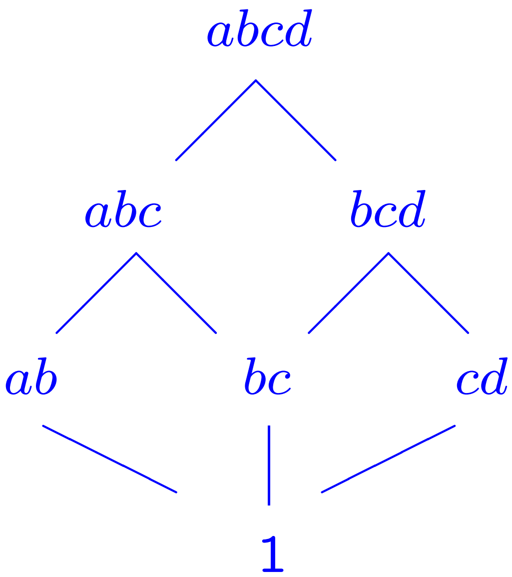

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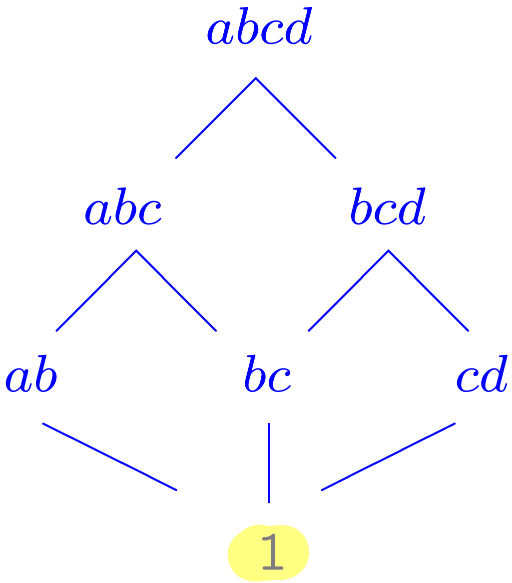


Gasharov - Peeva - Welker (1999): I monomial ideal with lcm lattice L

$$\beta_{i,\mathbf{m}} = \dim_k \widetilde{H}_{i-2}\left((1,\mathbf{m})_L; k\right)$$

Note that the monomials in the lcm lattice index the betti numbers

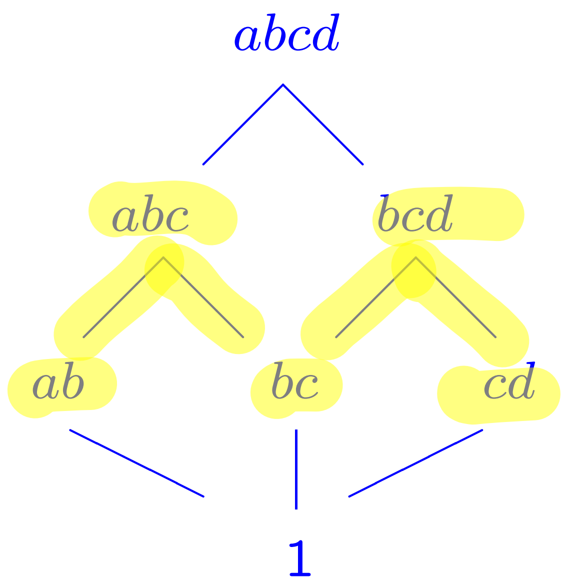
$$\beta_i = \sum_{m \in LCM(I)} \beta_{i,m}$$

The lcm lattice

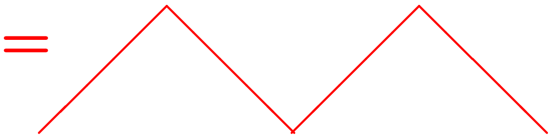
m = abcd

Example. $I = (ab, bc, cd)$

$\beta_{i,abcd} = 0 \quad \forall i$



$(1, abcd) =$



↓

dim = 1

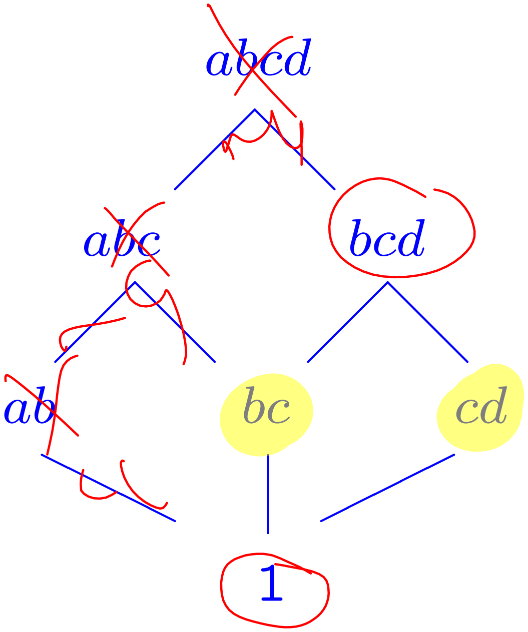
Gasharov - Peeva - Welker (1999): I monomial ideal with lcm lattice L

$$\beta_{i,\mathbf{m}} = \dim_k \widetilde{H}_{\underline{\underline{\underline{i-2}}}}((1,\mathbf{m})_L; k) = 0$$

The lcm lattice

$\beta_2(I) \neq 0$

Example. $I = (ab, bc, cd)$



$\beta_{2,bcd} = 1$

$(1, bcd) = \bullet \qquad \bullet$

Gasharov - Peeva - Welker (1999): I monomial ideal with lcm lattice L

$$\beta_{i,\mathbf{m}} = \dim_k \widetilde{H}_{i-2}((1,\mathbf{m})_L; k)$$

$i-2=0 \Longrightarrow i=2$

The lcm lattice

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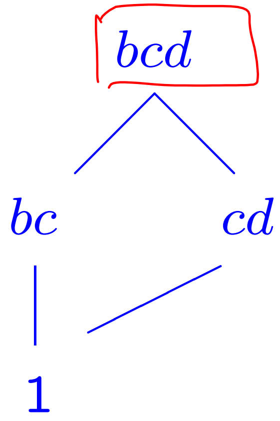

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$$\beta_{i,\mathbf{m}} = \dim_k \widetilde{H}_{i-2}((1, \mathbf{m})_L; k)$$

For a monomial ideal I

$\beta_{i,\mathbf{m}} \neq 0$ m some monomial

$\iff$ there is a non-acyclic lcm lattice L with maximal element $\mathbf{m}$

For a monomial ideal I

$\beta_{i,\mathbf{m}} \neq 0$

$\iff$ there is a non-acyclic lcm lattice L with maximal element $\mathbf{m}$

$\implies$ every proper element of L has a *complement* [Baclawski (1977)]

Two monomials $\mathbf{m}_1$ and $\mathbf{m}_2$ are *complements* if

$$\mathrm{lcm}(\mathbf{m}_1, \mathbf{m}_2) = \mathbf{m} \qquad \text{and} \qquad \mathrm{gcd}(\mathbf{m}_1, \mathbf{m}_2) \notin I$$

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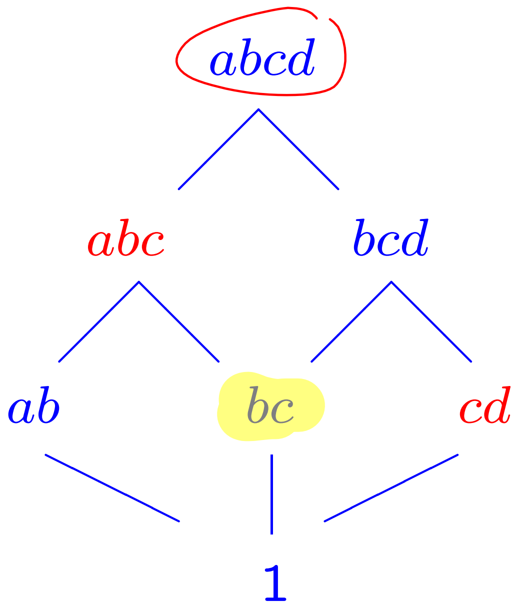

- abc and cd are complements

- bc has no complement

- $\boxed{\beta_{i,abcd} = 0}$

Subadditivity Conjecture:

$$\beta_i = \sum_{m \in LCM(I)} \beta_{i,m}$$

I monomial ideal

Fix i and let $\underline{t_i} = \max\{\deg m \mid \beta_{i,m} \neq 0\}$.

Subadditivity Property: $t_{a+b} \leq t_a + t_b$ for all $a, b. > 0$

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- *holds for some algebras of* $\dim \leq 1$ [Eisenbud - Huneke - Ulrich (2006)]

- *fails in general* [Avramov - Conca - Iyengar (2015)]

- *holds when* I *monomial ideal and* $a = 1$*, in cases where* b *is the projective dimension of* S/I *or when* I *is any monomial ideal* [Herzog - Srinivasan (2016)]

- *holds in certain homological degrees for Gorenstein algebras* [El Khoury - Srinivasan (2016)]

- *holds when* $a = 1, 2, 3$ *and* I *monomial ideal generated in degree 2* [Fernández-Ramos - Gimenez (2014), Abedelfatah - Nevo (2016)]

Subadditivity Conjecture:

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The question is open for the class of monomial ideals.

Subadditivity for monomial ideal I

Suppose L = lcm lattice of I has top monomial $\mathbf{m}$.

Question. If $\beta_{i,\mathbf{m}} \neq 0$ and $i = a + b$, are there complements $\mathbf{m}_1$ and $\mathbf{m}_2$ in L with nonzero multigraded Betti numbers $\beta_{a,\mathbf{m}_1} \neq 0$ and $\beta_{b,\mathbf{m}_2} \neq 0$?

Back to subadditivity for monomial ideal I

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If yes, then

1) $\text{lcm}(\mathbf{m}_1, \mathbf{m}_2) = \mathbf{m} \Longrightarrow \text{deg}(\mathbf{m}_1) + \text{deg}(\mathbf{m}_2) \geq \text{deg}(\mathbf{m})$

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2) Recall: $t_a = \max\{\deg \mathbf{m} \mid \beta_{a,\mathbf{m}} \neq 0\}$.

3) $t_i = \deg(\mathbf{m})$ and $t_a \geq \deg(\mathbf{m}_1)$ and $t_{\underset{b}{\cancel{a}}} \geq \deg(\mathbf{m}_2)$

Therefore we have subadditivity

$$\boxed{t_i \leq t_a + t_b}$$

Subadditivity for lattices

Suppose L = lcm lattice of I has top monomial $\mathbf{m}$.

Question. L = lattice with

$$\widetilde{H}_{a+b-2}\overset{\text{whole lattice}}{\boxed{((1,\mathbf{m})_L;k)}} \neq 0$$

are there complements $\mathbf{m}_1$ and $\mathbf{m}_2$ in L with

$$\widetilde{H}_{a-2}((1,\mathbf{m}_1)_L;k) \neq 0 \text{ and } \widetilde{H}_{b-2}((1,\mathbf{m}_2)_L;k) \neq 0?$$

Theorem. (Faridi 2019) If I is the facet ideal of a simplicial forest, then the answer is positive, and therefore subadditivity holds.

Subadditivity in general

Question. If L is a lattice with

$$\widetilde{H}_{a+b-2}(L;k) \neq 0$$

are there induced sublattices $\underline{L}_1$ and $\underline{L}_2$ with

$$\widetilde{H}_{a-2}(L_1;k) \neq 0 \text{ and } \widetilde{H}_{b-2}(L_2;k) \neq 0? \qquad + \text{ complements}$$

$\cong$ **Question.** If Γ is a simplicial complex on the vertex set $\{x_1,\ldots,x_n\}$ with

$$\widetilde{H}_{n-a-b-1}(\Gamma) \neq 0$$

are there $C, D \subseteq \{x_1,\ldots,x_n\}$ such that

$$C \cup D = \{x_1,\ldots,x_n\} \text{ and } C \cap D \in \Gamma;$$

$$\widetilde{H}_{\underbrace{|C|-a-1}}(\Gamma_C) \neq 0 \text{ and } \widetilde{H}_{\underbrace{|D|-b-1}}(\Gamma_D) \neq 0?$$

Γ_I: Stanley-Reisner complex of I

Hochster's formula: $\beta_{i,m} = \dim_k \widetilde{H}_{\deg m - i - 1}(\Gamma_m)$

Subadditivity for Simplicial complexes

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$$\widetilde{H}_{|C|-a-1}(\Gamma_C) \neq 0 \text{ and } \widetilde{H}_{|D|-b-1}(\Gamma_D) \neq 0?$$

Theorem. (Faridi-Shahada 2020) Yes, when $n-a-b-1$ is the smallest size of a nonface of Γ.

More cases: work in progress (almost there....?)

Bjorner "Topological methods", Handbook of combinatorics (vol. 2)March 1996 Pages 1819–1872

No algebra, ? just topology!

Subadditivity for Simplicial complexes

Question. $\widetilde{H}_{n-a-b-1}(\Gamma) \neq 0$ are there $C, D \subseteq \{x_1, \ldots, x_n\}$ such that

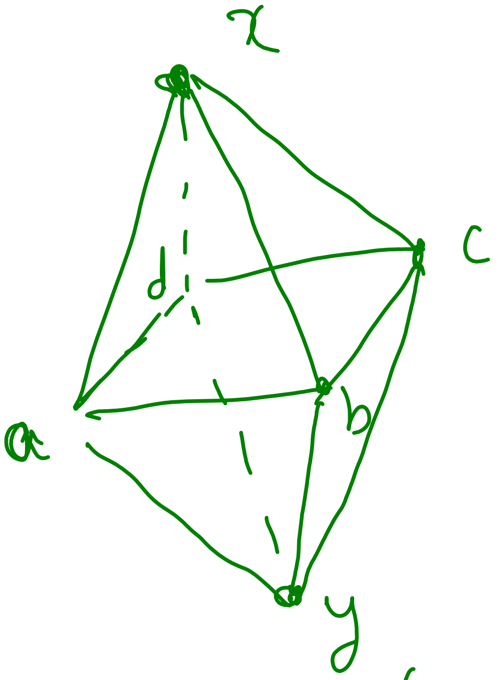
$C \cup D = \{x_1, \ldots, x_n\}$ ✓ and $C \cap D \in \Gamma$, and

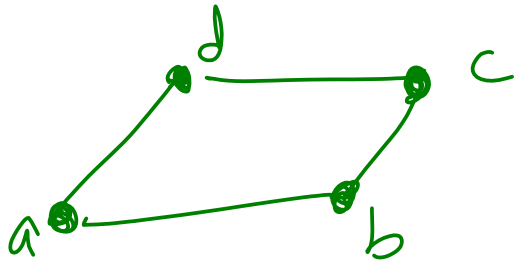
$\widetilde{H}_{|C|-a-1}(\Gamma_C) \neq 0$ and $\widetilde{H}_{|D|-b-1}(\Gamma_D) \neq 0$?

$C \cap D = \emptyset$

$D = \{a, b, c, d\}$

$C = \{x, y\}$





$\widetilde{H}_0(\Gamma_C) \neq 0$

$|C| - \underline{a} - 1 = 0$

$\Rightarrow \underline{a} = 2 - 1 = 1$ ✓

$\widetilde{H}_1(\Gamma_D) \neq 0$

$|D| - \underline{b} - 1 = 1$

$\underline{b} = 4 - 2 = 2$ ✓

$\widetilde{H}_2 \neq 0 \Rightarrow \overset{6}{\cancel{n}} - \underline{a} - \underline{b} - 1 = 2 \quad \underline{a} + \underline{b} = 3 \quad \underline{a} = 1 \quad \underline{b} = 2$