Stone Duality for Topological Convexity Spaces

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Section 1

Topological Convexity Spaces

Abstract Convexity Spaces

Definition

A topological convexity space is $(X, \mathcal{F}, \mathcal{C})$ where $\mathcal{C}, \mathcal{F} \subseteq PX$, where

- $\mathcal{C}$ is closed under arbitrary intersections and directed unions
- F is closed under arbitrary intersections and finite unions.

Definition

If $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ are topological convexity spaces, a function $f : X \longrightarrow X'$ is a homomorphism of topological convexity spaces if:

- for any $A \in \mathcal{F}'$, $f^{-1}(A) \in \mathcal{F}$ and
- for any $C \in \mathcal{C}'$, $f^{-1}(C) \in \mathcal{C}$.

Compatibility between Topology and Convexity

Definition

A topological convexity space $(X, \mathcal{F}, \mathcal{C})$ is compatible if the following conditions hold:

1. All convex sets are connected.
2. All finitely generated convex sets are closed and compact.
3. Convex closure preserves closed sets.

Examples

Metric Spaces

Let (X, d) be a metric space. We define a topological convexity space structure on X by

- $\mathcal{F} = \{A \subseteq X | (\forall x \in X) \left(\bigwedge\{d(x,y) | y \in A\} = 0 \Rightarrow x \in A\right)\}$
- $\mathcal{C} = \{C \subseteq X | (\forall x, z \in C)(d(x,y) + d(y,z) = d(x,z) \Rightarrow y \in C)\}$

Complete Lattices

Let $(L, \bigvee, \bigwedge)$ be a complete lattice. We can define a topological convexity space on L by

- $\mathcal{F} = \{\text{Intersections of finitely generated downsets}\}$
- $\mathcal{C} = \{\text{Ideals}\}$

Lattices are Compatible

Proposition

For a lattice L, the topological convexity space $(L, \mathcal{S}, \mathcal{I})$ is compatible, where $\mathcal{F}$ is the set of intersections of finitely-generated downsets (Scott-closed sets) and $\mathcal{I}$ is the set of ideals.

Proof.

1. If $p \in I \setminus S$ and $q \in I \setminus T$, then $p \vee q \in I \setminus (S \cup T)$. Thus if $I \subseteq S \cup T$ then $I \subseteq S$ or $I \subseteq T$.
2. Finitely generated ideals principal — thus totally compact.
3. Ideal generated by Scott-closed subset, closed under arbitrary joins, so principal.

□

Sup lattices

Lemma

$L \xrightarrow{f} M$ is a sup-homomorphism if and only if f is a homomorphism of topological convexity spaces.

Proof ($\Rightarrow$).

- f is order preserving, so f^{-1} preserves downsets.
- If $a, b \in f^{-1}(I)$ for an ideal I, then $f(a \vee b) = f(a) \vee f(b) \in I$, so $f^{-1}(I)$ is an ideal.
- If F is a finitely-generated downset in M, generated by elements $x_1, \ldots, x_n \in M$, then $f^{-1}(F)$ is generated by $f^*(x_1), \ldots, f^*(x_n)$ where f^* is the adjoint to f.

□

Sup lattices

Lemma

$L \xrightarrow{f} M$ is a sup-homomorphism if and only if f is a homomorphism of topological convexity spaces.

Proof ($\Leftarrow$).

- f is order preserving — if $a \leqslant b$, then $a \in f^{-1}(\downarrow(f(b)))$.
- Let $A \in DL$ and let $a = \bigvee A$.
- Let $x = \bigvee\{f(b)|b \in A\}$. $f^{-1}(\downarrow(x))$ is a principal ideal $\downarrow(c)$.
- For any $b \in A$, $f(b) \in \downarrow(x)$, so $b \leqslant c$. It follows that $a \leqslant c$.
- Conversely, $(\forall b \in A)(f(a) \geqslant f(b))$, so $f(a) \geqslant x$ and $f(a) \leqslant f(c) \leqslant x$.

□

Section 2

Stone Duality for Topological Convexity Spaces

Reflection

Proposition

The inclusion $\underline{\mathcal{S}\mathbf{up}} \rightarrowtail \underline{\mathcal{TC}}$ *is reflective.*

Proof.

The reflection function sends a topological convexity space $(X, \mathcal{F}, \mathcal{C})$ to the lattice $\mathcal{F} \cap \mathcal{C}$ ordered by inclusion.

- It is easy to see that this splits $\underline{\mathcal{S}\mathbf{up}} \rightarrowtail \underline{\mathcal{TC}}$.
- For a topological convexity space $(X, \mathcal{F}, \mathcal{C})$, the unit η_X is given by $\eta_x(x) = \overline{\{x\}} = \bigcap\{T \in \mathcal{F} \cap \mathcal{C} | x \in T\}$.

□

Preconvexity Spaces, Distributive Partial Sup Lattices

Definition

A preconvexity space is $(X, \mathcal{P})$, where $\emptyset \in \mathcal{P} \subseteq PX$ is closed under arbitrary intersection.

Definition

A distributive partial sup lattice is an object X of $\underline{\mathcal{I}\mathbf{nf}}$ with a partial homomorphism $DX \xrightarrow{\bigvee} X$ satisfying:

1. For $a \in A \in DX$, $a \leqslant \bigvee A$ when defined.
2. $\bigvee(\downarrow(a)) = a$ for all $A \in X$.
3. Every $\bigvee^{-1}(x)$ is an interval in DX.

Proposition

$$\underline{\mathcal{TC}} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \underline{\mathcal{P}\mathbf{rec}} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \underline{\mathcal{D}\mathbf{ist}\mathcal{P}\mathbf{art}\mathcal{S}\mathbf{up}} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \underline{\mathcal{S}\mathbf{up}}$$

$$(X, \mathcal{F}, \mathcal{C}) \longmapsto (X, \mathcal{F} \cap \mathcal{C}) \longmapsto (\mathcal{F} \cap \mathcal{C}, \bigcap, \bigcup) \longmapsto (\mathcal{F} \cap \mathcal{C}, \bigcap)$$

$$(L, \mathcal{F}, \mathcal{I}) \leftarrow\!\shortmid (L, \{\downarrow(x) | x \in L\}) \leftarrow\!\shortmid (L, \bigwedge, \{\downarrow(x) | x \in L\}) \longleftarrow\!\shortmid (L, \bigwedge)$$

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Extension of Stone Duality

Stone Duality

$\underline{\mathcal{T}\mathbf{op}}_{T_0} \longrightarrow \underline{\mathcal{C}\mathbf{oframe}}^{\mathrm{op}}$

- Restriction to spatial coframes is a faithful fibration.
- Fibres are partial orders.
- Adjoint is top element of each spatial fibre.

Extended Stone Duality

$\underline{\mathcal{TC}}_{T_0} \longrightarrow \underline{\mathcal{S}\mathbf{up}}$

- Faithful fibration.
- Fibres are partial orders.
- Adjoint is top element of each fibre.

Not an Extension

However, the Stone duality between $\underline{\mathcal{T}\mathbf{op}}$ and $\underline{\mathcal{C}\mathbf{oframe}}^{\mathrm{op}}$ is not a restriction of the duality between $\underline{\mathcal{TC}}$ and $\underline{\mathcal{S}\mathbf{up}}$.
If L is a coframe, the corresponding topological space consists of finitely indecomposable points, not all points.

Fibres are Partially Ordered

Proposition

If $(X,\mathcal{F},\mathcal{C})$ and $(X',\mathcal{F}',\mathcal{C}')$ are T_0 topological convexity spaces in the same fibre then any parallel pair of vertical homomorphisms $X \underset{g}{\overset{f}{\rightrightarrows}} X'$ has $f=g$.

Proof.

- f, g vertical means $f^{-1}=g^{-1}$.
- For $x \in X$, let $a=f(x)$, $b=g(x)$.
- For any $a \in A \in \mathcal{F}' \cap \mathcal{C}'$, $f^{-1}(A)=g^{-1}(A)$.
- Since $x \in g^{-1}(A)$, $b \in A$. Conversely $b \in A \Rightarrow a \in A$.
- Since X is T_0, $(\forall A \in \mathcal{F}' \cap \mathcal{C}')(a \in A \Leftrightarrow b \in A)$ means $a=b$. □

T_D Spaces

Definition

A topological space X is T_D if for every $x \in X$, $\{x\}$ is an open subset of $\overline{\{x\}}$.

Proposition

T_D spaces are bottom elements in fibres over their coframe of closed sets.

Definition

A preconvexity space X is T_D if for every $x \in X$ $\overline{\{x\}} \setminus \{x\}$ is closed convex.

Proposition

Indecomposable elements of L form a preconvexity space where preconvex sets are prinicipal downsets of L.

Extension of T_D Spaces

Proposition

If L is a coframe join-generated by join-indecomposable elements, then the collection I of indecomposable elements of L with the collection $\{\downarrow(x) \cap I | x \in L\}$ is a T_D topological space.

Proof.

- Clearly, $\bigcap_{i\in I}(\downarrow(x_i) \cap I) = \downarrow\left(\bigwedge_{i\in I} x_i\right) \cap I$.
- Let $x, y \in I$ and $z \in I$ with $z \leqslant x \vee y$.
- Since L is a coframe, $(z \wedge x) \vee (z \wedge y) = z \wedge (x \vee y) = z$.
- Since z is indecomposable, it follows that $z \leqslant x$ or $z \leqslant y$.
- Hence $(x \cap I) \cup (y \cap I) = (x \vee y) \cap I$, $\Rightarrow$ topological space.

□

Equivalent ways to Specify a Preconvexity Space

Completely Distributive Partial Sup Lattices

Complete lattice X with a partial inf-homomorphism $DX \rightharpoonup X$ which is a restriction of the supremum function.

Lattices with Totally Below relation

Complete lattice X with preorder $\ll$ on X such that every $x \in X$ is the join of $\Downarrow(x)$.

Dense embeddings of coframes into completely distributive lattices

Dense inf-homomorphism $X \xrightarrow{i} Y$ where X is a coframe and Y is completely distributive.

Section 3

The Category of Topological Convexity Spaces

Darboux functions

Definition

A homomorphism $X \xrightarrow{f} Y$ of topological convexity spaces, is Darboux if the forward image of a convex set is convex.

Proposition

A sup-homomorphism $L \xrightarrow{f} M$ is a Darboux homomorphism between topological convexity spaces iff $(\exists t \in M)(ff^* = t \wedge \_)$.

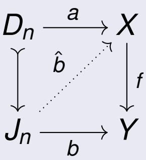
Proof.

- Darboux says that the forward image of an ideal is an ideal.
- Let $t = f(\top)$. Since $f(L)$ is an ideal, and $t \in f(L)$, for any $a \in M$, $a \wedge t \leqslant t$, so $a \wedge t \in f(L)$. Let $a \wedge t = f(x)$.
- Since $f(x) \leqslant a$, $x \leqslant f^*(a)$, so $ff^*(a) \geqslant f(x) = a \wedge t$. □

Internal Characterisation of Darboux Homomorphisms

Proposition

A $X \xrightarrow{f} Y$ in $\underline{\mathcal{TC}}$ is Darboux iff we can always find a diagonal factorisation in the commutative square



Proof.

- Must define $\hat{b}(i) = a(i)$ for $i = 1, \ldots, n$
- Since f is Darboux, $b(0)$ is in image of $\overline{a(D_n)}$.

□

Internal Characterisation of Darboux Homomorphisms

Proposition

A $X \xrightarrow{f} Y$ in $\underline{\mathcal{TC}}$ is Darboux iff we can always find a diagonal factorisation in the commutative square

$$\begin{array}{ccc} D_n & \xrightarrow{a} & X \\ \downarrow & \overset{\hat{b}}{\nearrow} & \downarrow f \\ J_n & \xrightarrow[b]{} & Y \end{array}$$

Proof.

- If $y \in \overline{f(A)}$, finite subset $D_n \xrightarrow{s} X$ has $y \in \overline{f(S)}$.
- This gives a commutative square $fs = ti$ where $t(0) = y$.
- Extension means that $y = f(v)$ for some $v \in \overline{S}$. □

Internal Characterisation of Darboux Homomorphisms

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Corollary

Darboux homomorphisms are closed under pullback in $\underline{\mathcal{TC}}$.

Proof.

Pullback property induces diagonal factorisation. □

Examples of Darboux Quotients

Proposition

If $\mathbb{R}^n \xrightarrow{f} Q$ is Darboux and strong epi, then $Q \cong \mathbb{R}^k$ for some k. If $k > 1$ then f is linear, and if $k = 1$ then f factors through a linear function with codomain $\mathbb{R}$.

Proof.

- For any $q^{-1}(a)$, $a \in Q$, and any tangent line l, for any $d \in q(l)$, $q^{-1}(d)$ must lie on the same side of l as $q^{-1}(a)$.
- If $x, y \in q^{-1}(b)$, if tangents from x, y to $q^{-1}(a)$ exist, they must be parallel.
- Only possible if either q factors through linear $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$, or for any line l and $a \in Q$, $(l \cap q^{-1}(a)) \in \{\emptyset, \{z\}, l\}$. □

Parallel Quotients

Definition

A homomorphism $X \overset{q}{\Longrightarrow\!\!\!\circ\!\!\!\Longrightarrow} Q$ in $\mathcal{D}arboux$ is a parallel quotient if for any $1 \xrightarrow{a} Q$, the pullback

$$\begin{array}{ccc} A & \rightarrowtail & X \\ \downarrow & & \Downarrow q \\ 1 & \xrightarrow[a]{} & Q \end{array}$$

is also a pushout.

Examples of Parallel Quotients

Proposition

A parallel quotient $\mathbb{R}^n \overset{q}{=\!\circ\!\Rightarrow} Q$ *is a linear function* $\mathbb{R}^n \xrightarrow{q} \mathbb{R}^m$.

Proof.

- If q is not linear, it factors as $\mathbb{R}^n \xrightarrow{r} \mathbb{R} \xrightarrow{s} \mathbb{R}$ where r is linear and s is monotone continuous.
- For this factorisation, s is also a parallel quotient. Therefore, it is sufficient to prove the result for $n = 1$.
- For $[a, b] \subseteq \mathbb{R}$, the pushout in $\mathcal{D}arboux$ is

$$\begin{array}{ccc} [a,b] & \rightarrowtail & \mathbb{R} \\ \downarrow & & \Big\downarrow{\scriptstyle\circ}\, f \\ 1 & \longrightarrow & \mathbb{R} \end{array} \quad \text{where } f(x) = \begin{cases} x & \text{if } x < a \\ a & \text{if } a \leqslant x \leqslant b \\ x + a - b & \text{if } x > b \end{cases}$$

□

Half-Spaces

Definition

A half-space is a convex set with convex complement.

Lemma

Closed half-spaces $H \rightarrowtail X$ correspond exactly to pullbacks

$$\begin{array}{ccc} H & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \xrightarrow[\chi_H]{} & S \end{array}$$

where $S = (2, \{\emptyset, \{1\}, 2\}, P(2))$ is the "Sierpinski" space.

Proof.

$\{0\}$ and $\{1\}$ convex $\Rightarrow H = (\chi_H)^{-1}(1)$, $H^c = (\chi_H)^{-1}(0)$ both convex. $\{1\}$ closed $\Rightarrow H$ closed. $\square$

Kakutani Spaces

Definition

$X \in \mathrm{ob}(\underline{\mathcal{TC}})$ is Kakutani if for every $A, B \in \mathcal{F} \cap \mathcal{C}$ with $A \cap B = \emptyset$, there is a closed half-space H with $A \subseteq H$ and $B \cap H = \emptyset$.

Remarks

- The traditional definition does not mention the topology.
- From the previous lemma, we can consider χ_H so $\chi_H(A) = \{0\}$ and $\chi_H(B) = \{1\}$.
- We can define this as a lifting property

$$\begin{array}{ccccc} & & 4 & \longrightarrow & S \\ & \nearrow & \downarrow & & \\ X & \xrightarrow[\chi_{A,B}]{} & 3 & & \end{array}$$

Section 4

Future work

Euclidean Spaces (Next time)

Axiomatising Euclidean Spaces in $\underline{\mathcal{TC}}$

- Most Darboux quotients are parallel.
- Strong Kakutani-type properties.
- Standard topological properties - Connected, T_1, etc.

Convexity Manifolds

Example (Projective Spaces)

- The projective space $\mathbb{P}^n$ is not a convexity space, because there is not a global convexity structure.
- However, if we choose which hyperplane is at infinity, there is a unique convexity structure.
- Thus, projective spaces should be some kind of convexity manifold.

Metrics and Measures

Example (Σ-algebras)

- Let $(X, \mathcal{B})$ be a Σ-algebra.
- Define a topological convexity space $(\mathcal{B}, \mathcal{F}, \mathcal{I})$ where $\mathcal{I}$ is the set of intervals in the lattice $\mathcal{B}$ and $\mathcal{F}$ is the set of collections of measureable sets closed under limits of characteristic functions.
- Metrics d inducing this topological convexity space on $\mathcal{B}$ are of the form $d(A, B) = \mu(A \triangle B)$, where μ is a measure on $(X, \mathcal{B})$.