Euclidean Spaces in the Category of Topological Convexity Spaces

Toby Kenney

Dalhousie University

ATCAT seminar
19-01-2021

Section 1

Introduction & Axioms

Previous Talk

- $\underline{\mathcal{TC}}$ — Category of topological convexity spaces.
- Homomorphisms — inverse image preserves closed sets, convex sets.
- Stone Duality between $\underline{\mathcal{TC}}$ and $\underline{\mathcal{S}\mathbf{up}}$.
- Darboux homomorphisms — forward image preserves convex sets.
- Parallel quotient — occur as pushout of any pullback in $\mathcal{D}arboux$.
- Half-space — convex set with convex complement.
- Kakutani space — disjoint closed convex sets separated by closed half-space.

Axioms for Euclidean Space

Definition

A Euclidean convexity space is a topological convexity space E satisfying the following axioms:

Axioms (Basic Topological Properties)

Axiom 1 *E is connected.*

Axiom 2 *Singletons of E are closed and convex.*

Axiom 3 *Convex subsets of E are directed unions of closed convex subsets of E, and closed subsets of E are intersections of finite unions of closed convex subsets of E.*

Axioms for Euclidean Space

Definition

A Euclidean convexity space is a topological convexity space E satisfying the following axioms:

Axioms (Half-Spaces)

Axiom 4 *Any Darboux quotient of E is Kakutani*

Axiom 5 *For any half-space $H \subseteq E$, , there is a pushout*

$$\begin{array}{ccc} H & \rightarrowtail & E \\ \downarrow & & \downarrow{\scriptstyle \circ}\, q_H \\ 1 & \rightarrowtail & Q_H \end{array}$$

in $\mathcal{D}$arboux. For this pushout, $H = {q_H}^{-1}(q_H(H))$.

Axioms for Euclidean Space

Definition

A Euclidean convexity space is a topological convexity space E satisfying the following axioms:

Axioms (Enough Parallel Quotients)

Axiom 6

(a) *For every* $E \xrightarrow{f} X$, *there is a universal* $E \overset{q}{\Longrightarrow} Q$ *factoring through* f.

(b) *General position* $G \rightarrowtail E$, *any* $G \xrightarrow{g} R$,

$$\begin{array}{ccc} G & \rightarrowtail & E \\ {\scriptstyle g}\downarrow & & \Downarrow{\scriptstyle q} \\ H & \rightarrowtail & Q \end{array}$$

Axioms for Euclidean Space

Definition

A Euclidean convexity space is a topological convexity space E satisfying the following axioms:

Axioms (Lines and Dimensions)

Axiom 7 *For any* $J_2 \overset{m}{\rightarrowtail} E$*, in the parallel pushout,*

$$\begin{array}{ccc} J_2 & \xrightarrow{j} & D_2 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ E & \Longrightarrow & Q \end{array}$$

the morphism d *factors through* 1*.*

Axiom 8 *General position monomorphism* $D_4 \rightarrowtail E$*.*

Section 2

Affine Spaces and Lines

Affine Spaces

Definition

An affine space is a subspace $A \subseteq E$ such that $A \longrightarrow 1$ is the pullback of a parallel quotient:

$$\begin{array}{ccc} A & \rightarrowtail & E \\ \downarrow & & \Downarrow \\ 1 & \longrightarrow & Q \end{array}$$

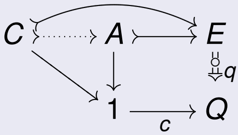
Affine Closure

Proposition

For closed convex $C \subseteq E$, $\exists$ smallest affine space containing C.

Proof (sketch).

- Take the pullback of the parallel pushout:



□

Affine Closure

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Proof (sketch).

- Now we have:

$$C \rightarrowtail A' \rightarrowtail E$$

□

Affine Closure

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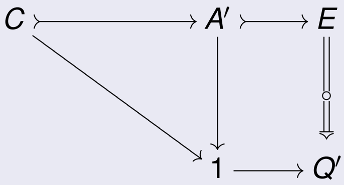
Proof (sketch).

- Since A' is affine:



Affine Closure

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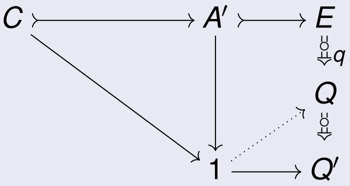
Proof (sketch).

- By the pushout property:



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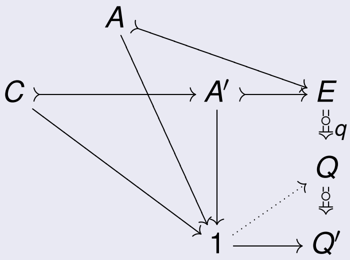

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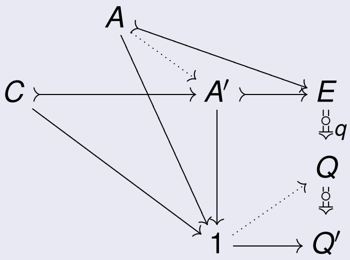

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Affine Closure

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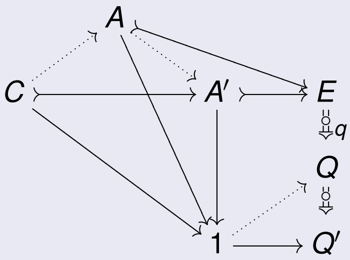


□

Affine Spaces and Half-spaces

Proposition

H closed half-space, A affine space & $A \subseteq H$ or $A \subseteq H^c$, then the pushout q_H from Axiom 5 factors through the pushout q_A.

Proof (sketch).

- In the pushout from Axiom 5, the space Q_H is Kakutani and the singleton $\{q_H(H)\}$ is a half-space.

$$\begin{array}{ccc} H & \rightarrowtail & E \\ \downarrow & & \big\downarrow{\circ}\, q_H \\ 1 & \rightarrowtail & Q_H \end{array}$$

□

Affine Spaces and Half-spaces

Proposition

H closed half-space, A affine space & $A \subseteq H$ or $A \subseteq H^c$, then the pushout q_H from Axiom 5 factors through the pushout q_A.

Proof (sketch).

- In the pushout from Axiom 5, the space Q_H is Kakutani and the singleton $\{q_H(H)\}$ is a half-space.
- If $A \subseteq H$ then

$$\begin{array}{ccccc} A & \rightarrowtail & H & \rightarrowtail & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle q_H} \\ 1 & \longrightarrow & 1 & \rightarrowtail & Q_H \end{array}$$

factors through the pushout

$$\begin{array}{ccc} A & \rightarrowtail & E \\ \downarrow & & \downarrow{\scriptstyle q_A} \\ 1 & \rightarrowtail & Q_H \end{array}$$

□

Affine Spaces and Half-spaces

Proposition

H closed half-space, A affine space & $A \subseteq H$ or $A \subseteq H^c$, then the pushout q_H from Axiom 5 factors through the pushout q_A.

Proof (sketch).

- In the pushout from Axiom 5, the space Q_H is Kakutani and the singleton $\{q_H(H)\}$ is a half-space.
- $A \subseteq H^c \Rightarrow (\exists K)(A \subseteq K \subseteq H^c)$. (Kakutani property)
- $(\forall x \in E)({q_A}^{-1}(q_A(x)) \subseteq {q_K}^{-1}(q_K(x)) \subseteq K \vee \{x\})$.
- For $x \notin H$, we can show $K \vee \{x\} \subseteq H^c$. Thus $q_A(H) \cap q_A(H^c) = \emptyset$.

□

Lines

Definition

For any $x, y \in E$, the line $\langle x, y \rangle$ is defined as either:

(i) $\langle x, y \rangle = \bigcap \{A \subseteq E | x, y \in A, A \text{ is affine}\}$

(ii) $\langle x, y \rangle = \{z \in E | z \in [x, y] \text{ or } x \in [y, z] \text{ or } y \in [x, z]\}$

Lines

Theorem

The two definitions of lines are equivalent.

Proof.

(i)⊆(ii) If x, y, z in general position, By Axiom 6, we have

$$\begin{array}{ccc} D_3 & \xrightarrow{x,y,z} & E \\ \downarrow & & \Downarrow q \\ D_2 & \longrightarrow & Q \end{array}$$

Now, $q(x) = q(y) \neq q(z)$, so $q^{-1}(q(x))$ is an affine space containing x and y but not z.

□

Lines

Theorem

The two definitions of lines are equivalent.

Proof.

(ii)⊆(i) If $y \in [x, z]$, then $J_2 \overset{x,y,z}{\rightarrowtail} E$ is Darboux. By Axiom 7,

$$\begin{array}{ccc} J_2 & \overset{x,y,z}{\rightarrowtail} & E \\ \downarrow & & \Downarrow q \\ D_2 & \longrightarrow & 1 \end{array}$$

is a parallel pushout. Thus any $E \overset{r}{\Longrightarrow} R$, with $r(x) = r(y)$, factors through q, so $r(x) = r(z)$.

□

Parallel Lines & Parallellograms

Definition

The lines $\langle a, b\rangle$ and $\langle c, d\rangle$ are parallel if for any parallel quotient $E \overset{q}{=\!\circ\!\Rightarrow} Q$, $q(a) = q(b)$ if and only if $q(c) = q(d)$.

Proposition

For any $a, b, c \in E$, there is a unique line $\langle c, d\rangle$ parallel to $\langle a, b\rangle$.

Sketch proof.

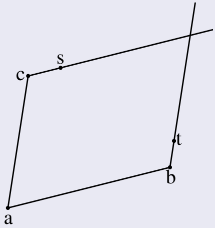
For any parallel quotient $E \overset{q}{=\!\circ\!\Rightarrow} Q$ such that $q(a) = q(b)$, there is an affine space $A_q = q^{-1}(q(c))$. The intersection of all these spaces is a line $\langle c, d\rangle$. □

Parallellograms

Proposition

For any a, b, c in general position, there is a unique $d \in E$ such that $\langle b, d \rangle$ is parallel to $\langle a, c \rangle$ and $\langle c, d \rangle$ is parallel to $\langle a, b \rangle$.

Proof.



Let $\langle c, s \rangle$ parallel to $\langle a, b \rangle$ and $\langle b, t \rangle$ parallel to $\langle a, c \rangle$.
If $\langle c, s \rangle \cap \langle b, t \rangle = \emptyset$, $\exists$ half-spaces $H \supseteq \langle c, s \rangle$, $K \supseteq \langle b, t \rangle$, and $H \cap K = \emptyset$.
H and K are parallel to both $\langle c, s \rangle$ and $\langle b, t \rangle$. Therefore they are parallel to $\langle a, b, c \rangle$. □

Modular Lattice of Affine Spaces

Lemma

If $p, q \in \langle a, b, c\rangle$ and a, p, q in general position, then $\langle a, p, q\rangle = \langle a, b, c\rangle$.

Proof.

- For $s \in \langle a, b, c\rangle$, the line through s parallel to $\langle b, c\rangle$ intersects both $\langle a, b\rangle$ and $\langle a, c\rangle$.
- If $s \in \langle a, b, c\rangle$ and $s \notin \langle a, b\rangle$ then $\langle a, b, s\rangle = \langle a, b, c\rangle$.

□

Lines in Two Dimensions

Proposition

If l and m are two lines in $\langle a, b, c\rangle$, then either l and m are parallel, or $l \cap m$ is a singleton.

Sketch proof.

- Let $\langle a, p\rangle$ parallel to l and $\langle a, q\rangle$ parallel to m.
- If l, m not parallel, then $\langle a, p, q\rangle$ in general position.
- If l, m disjoint, $\exists$ closed half-space H separating them.
- Affine space parallel to ∂H through a must contain p and q.
- Since $\langle a, p\rangle$ is parallel to $l \subseteq \langle a, b, c\rangle$, we have $p \in \langle a, b, c\rangle$.
- Similarly, $q \in \langle a, b, c\rangle$.
- By previous lemma, $\langle a, p, q\rangle = \langle a, b, c\rangle$. □

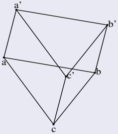
Desargue's Theorem

Theorem

If $\langle a, b, c\rangle$ are in general position and

- $\langle a, b\rangle$ *is parallel to* $\langle a', b'\rangle$
- $\langle a, c\rangle$ *is parallel to* $\langle a', c'\rangle$
- $\langle a, a'\rangle$, $\langle b, b'\rangle$ *and* $\langle c, c'\rangle$ *are parallel*

Then $\langle b, c\rangle$ *is parallel to* $\langle b', c'\rangle$



Proof.

- If a, b, c, a' in general position, this follows from modularity of the affine lattice.
- If $a' \in \langle a, b, c\rangle$ then choose a'' not in $\langle a, b, c\rangle$
- Apply result for a, b, c, a'', b'', c'', and for $a'', b'', c'', a', b', c'$.

□

Section 3

Translations and Vector Space Structure

Translations

Definition

A translation is an automorphism $E \xrightarrow{\tau} E$ such that:

- $(\forall a, b \in E)$, $\langle \tau(a), \tau(b) \rangle$ is parallel to $\langle a, b \rangle$
- the set of fixed points of τ is either empty or the whole of E.

Theorem

$\forall a, b \in E$, there is a unique translation τ_{ab} with $\tau_{ab}(a) = b$.

Sketch proof.

- For $x \notin \langle a, b \rangle$, $\tau_{ab}(x)$ is given by parallellogram completion.
- By Desargue's theorem, $\tau_{ab} = \tau_{x\tau_{ab}(x)}$
- Thus, if $x \in \langle a, b \rangle$ and $c \notin \langle a, b \rangle$, $\tau_{ab}(x) = \tau_{c\tau_{ab}(c)}(x)$. □

Abelian Group of Translations

Theorem

Translations of a Euclidean space form an abelian group.

Proof.

- $\langle \sigma\tau(x), \sigma\tau(y) \rangle$ parallel to $\langle \tau(x), \tau(y) \rangle$, parallel to $\langle x, y \rangle$.
- If $\sigma\tau(x) = x$ then $\tau = \tau_{x\tau(x)}$, and $\sigma = \tau_{\tau(x)x}$, so $\sigma\tau = 1_E$.
- $\langle \tau^{-1}(x), \tau^{-1}(y) \rangle$ is parallel to $\langle \tau\tau^{-1}(x), \tau\tau^{-1}(y) \rangle = \langle x, y \rangle$
- If $c \notin \langle a, b \rangle$ and $d = \tau_{ab}(c) = \tau_{ac}(b)$, $\tau_{ab}\tau_{ac} = \tau_{ad} = \tau_{ac}\tau_{ab}$.

□

$\mathbb{Z}$-module

Theorem

For $n \in \mathbb{N}$, the endofunction of E given by $e_n(x) = (\tau_{ax})^n(a)$ is a homomorphism of topological convexity spaces.

Theorem

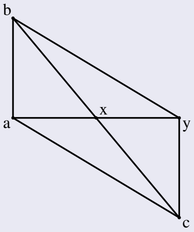
The homomorphism e_n has the property that for any $x, y \in E$, $\langle e_n(x), e_n(y) \rangle$ is parallel to $\langle x, y \rangle$

$\mathbb{Q}_2$-Module structure

Theorem

The homomorphism $E \xrightarrow{e_2} E$ *is invertible.*

Proof.



- *abyc* is a parallellogram.

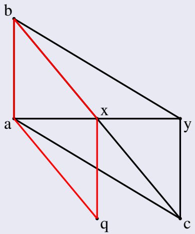
□

$\mathbb{Q}_2$-Module structure

Theorem

The homomorphism $E \xrightarrow{e_2} E$ is invertible.

Proof.



- *abyc* is a parallellogram.
- $q = \tau_{ba}(x)$, so $\tau_{aq} = \tau_{bx}$

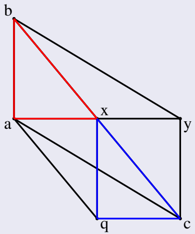
□

$\mathbb{Q}_2$-Module structure

Theorem

The homomorphism $E \xrightarrow{e_2} E$ is invertible.

Proof.



- *abyc* is a parallellogram.
- $q = \tau_{ba}(x)$, so $\tau_{aq} = \tau_{bx}$
- By Desargue's theorem, $\langle a, x\rangle$ and $\langle q, c\rangle$ are parallel.

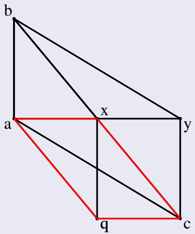
□

$\mathbb{Q}_2$-Module structure

Theorem

The homomorphism $E \xrightarrow{e_2} E$ is invertible.

Proof.



- *abyc* is a parallellogram.
- $q = \tau_{ba}(x)$, so $\tau_{aq} = \tau_{bx}$
- By Desargue's theorem, $\langle a, x\rangle$ and $\langle q, c\rangle$ are parallel.
- So $\tau_{bx}(x) = \tau_{aq}(x) = c$
- and $\tau_{ax}(x) = \tau_{qc}(x) = y$

□

$\mathbb{R}$-action

Theorem

For any $\lambda \in \mathbb{R}$, and any $x \in E$, there is a unique element y such that $[a, y] = \bigvee_{r \in \mathbb{Q}_2 | r \in [0,\lambda]} [a, e_r(x)]$

Definition

For $\lambda \in \mathbb{R}$, we define $e_\lambda(x)$ is the element such that $[a, \lambda * x] = \bigvee_{r \in \mathbb{Q}_2 | r \in [0,\lambda]} [a, e_r(x)]$

Theorem

The function e_λ is a homomorphism for any $\lambda \in \mathbb{R}$.

Topology

Theorem

A Euclidean space is a real vector space with product topology.

Proof.

- Every open set is a union of finite intersections of open half-spaces.
- Finite intersections of open half-spaces are open in the product topology.
- Conversely, every open set in the product topology is a union of finite intersections of open half-spaces.

□

Section 4

Future Work

Future work

Rewrite axioms in terms of the lattice of closed convex sets.

- T_1 spaces determined by lattice of closed convex sets.
- Can rewrite Euclidean axioms in terms of this lattice.

Euclidean-like spaces

- Let $\pi = (B_1, \ldots, B_k)$ be a partition of n.
- $S_n \xrightarrow{t_\pi} \prod_{i=1}^{k} S_{B_i}$, where $(t_\pi(\sigma))_i(w) = |\{j \in B_i | \sigma(j) < \sigma(w)\}|$ behaves very like a parallel quotient.
- Weaken Euclidean axioms to include this case.
- How much geometry remains for the weaker axioms?

Future Work (Cont.)

Algebraic Geometry

- Can define conic sections using only linear properties.
- Develop similar definitions for higher degree.
- What are these in Euclidean-like spaces?
- Can these generate Euclidean embeddings?

Sheaves on Topological Convexity Spaces?

- Extend sheaves to topological convexity spaces
- Sheaves based on open sets. Need to redefine.
- Fundamental notion is inequality.