Morphisms of rings

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Ring

- The category of rings **Ring** has rings with 1 as objects and homomorphisms preserving 1 as morphisms

- This is a very good category. It's monadic over **Set**, so complete and cocomplete. It's locally finitely presentable, so has a notion of finitely presentable object, and every object is a filtered colimit of them

- So why muck with it?

Bimodules

- Given rings R and S, an S-R-bimodule M is simultaneously a left S-module and a right R-module whose left and right actions commute

$$(sm)r = s(mr)$$

- If T is another ring and N a T-S-bimodule, the tensor product over S, $N \otimes_S M$ is naturally a T-R-bimodule. We have associativity isomorphisms

$$P \otimes_T (N \otimes_S M) \cong (P \otimes_T N) \otimes_S M$$

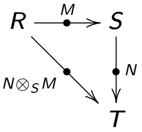
  and unit isomorphisms

$$M \otimes_R R \cong M \cong S \otimes_S M$$

- To keep track of the various rings involved and what's acting on what and on which side we can write

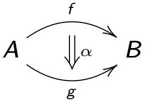
$$M : R \xrightarrow{\;\bullet\;} S$$

  to mean that M is an S-R-bimodule
- The tensor product looks like a composition



Bicategories

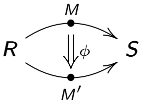
- Rings with bimodules as morphisms is not a category but a *bicategory*, $\mathcal{B}im$
- In a bicategory we have objects and morphisms which compose, but composition is only associative and unitary up to isomorphism
- To express this isomorphism we need morphisms between morphisms



called 2-*cells*

$\mathcal{B}im$

- In our example $\mathcal{B}im$
  - Objects are rings
  - Morphisms (1-cells) are bimodules
  - A 2-cell



is a linear map of bimodules, i.e. a function such that

$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$$

$$\phi(sm) = s\phi(m)$$

$$\phi(mr) = \phi(m)r$$

- $\mathcal{B}im$ is a very good bicategory
  - Cartesian bicategory
  - Biclosed

$$\frac{\dfrac{M \longrightarrow N \obslash_T P}{N \otimes_S M \longrightarrow P}}{N \longrightarrow P \oslash_R M}$$

Double categories

- A *double category* $\mathbb{A}$ has objects (A, B, C, D below) and *two* kinds of morphism, strong, which we call *horizontal* (f, g below) and *weak*, or *vertical* (v, w below) These are related by a further kind of morphism, double cells as in

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\downarrow\bullet & \overset{\alpha}{\Rightarrow} & \bullet\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$$

- The horizontal arrows form a category $\mathbf{Hor}\mathbb{A}$ with composition denoted by juxtaposition and identities by 1_A. Cells can also be composed horizontally forming a category
- The vertical arrows compose to give a bicategory $\mathcal{V}ert\mathbb{A}$ whose 2-cells are the *globular cells* of $\mathbb{A}$, i.e. those with identities on the top and bottom

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle v}\downarrow\bullet & \Rightarrow & \bullet\downarrow{\scriptstyle w} \\ C & \xrightarrow[1_C]{} & C \end{array}$$

Vertical composition is denoted by $\bullet$ and vertical identities by id_A

Example

$\mathbb{R}\mathrm{el}$ has sets as objects and functions as horizontal arrows, so $\mathbf{Hor}\mathbb{R}\mathrm{el} = \mathbf{Set}$. A vertical arrow $R : X \longrightarrow\!\!\!\!\!\bullet\;\; Y$ is a relation between X and Y and there is a unique cell

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ R \big\downarrow\!\!\bullet & \Rightarrow & \bullet\!\!\big\downarrow R' \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

if (and only if) we have

$$\forall_{x,y}(x \sim_R y \Rightarrow f(x) \sim_{R'} g(y))$$

The double category $\mathbb{R}\text{ing}$

- Objects are rings (with 1)
- Horizontal arrows are homomorphisms (pres. 1)
- Vertical arrows are bimodules
- A double cell

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle M}\downarrow & \overset{\phi}{\Longrightarrow} & \downarrow{\scriptstyle M'} \\ S & \xrightarrow[g]{} & S' \end{array}$$

is a linear map in the sense that it preserves addition and is compatible with the actions

$$\phi(sm) = g(s)\phi(m)$$

$$\phi(mr) = \phi(m)f(r)$$

Companions

- Let $\mathbb{A}$ be a double category, $f : A \longrightarrow B$ a horizontal arrow, and $v : A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$ a vertical one in $\mathbb{A}$. We say that v is a *companion* of f if we are given cells, the *binding cells*, α and β, such that

$$
\begin{array}{ccccc}
A & \xrightarrow{1_A} & A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\,\bullet\!\downarrow & \alpha & \bullet\!\downarrow{\scriptstyle v} & \beta & \bullet\!\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B & \xrightarrow[1_B]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\,\bullet\!\downarrow & \mathrm{id}_f & \bullet\!\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B'
\end{array}
$$

$$
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle \mathrm{id}_A}\,\bullet\!\downarrow & \alpha & \bullet\!\downarrow{\scriptstyle v} \\
A & \xrightarrow{f} & B \\
{\scriptstyle v}\,\bullet\!\downarrow & \beta & \bullet\!\downarrow{\scriptstyle \mathrm{id}_B} \\
B & \xrightarrow[1_B]{} & B
\end{array}
\quad \cdot = \cdot \quad
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
{\scriptstyle v}\,\bullet\!\downarrow & 1_v & \bullet\!\downarrow{\scriptstyle v} \\
B & \xrightarrow[1_B]{} & B
\end{array}
$$

  Companions, when they exist, are unique up to isomorphism, and we use the notation f_* to denote a choice of companion for f

- In $\mathbb{R}\mathrm{el}$, every function $f\colon A \longrightarrow B$ has a companion, viz. its graph $Gr(f) \subseteq A \times B$

Companions in $\mathbb{R}\mathrm{ing}$

Proposition

(a) In $\mathbb{R}\mathrm{ing}$, every homomorphism $f : R \longrightarrow S$ has a companion, namely S considered as an S-R-bimodule with actions $\bullet$ given by

$$\begin{aligned} s' \bullet s &= s's \\ s \bullet r &= sf(r) \end{aligned}$$

(b) A bimodule $M : R \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\!\longrightarrow S$ is a companion, i.e. is of the form f_* for some horizontal arrow f, if and only if it is free on one generator as a left S-module
(c) f is unique up to conjugation by a unit of S

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If $M : R \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow S$ is free as a left S-module with generator $m_0 \in M$. Then for every $r \in R$ there is a unique element $f(r) \in S$ such that

$$m_0 r = f(r) m_0$$

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If n_0 is another free generator, then there exists an invertible element $a \in S$ such that $n_0 = am_0$. If $g : R \longrightarrow S$ corresponds to n_0, we have

$$g(r)n_0 = n_0 r = am_0 r = af(r)m_0 = af(r)a^{-1}n_0$$

The 2-category $\mathcal{H}or\mathbb{A}$

- Every double category $\mathbb{A}$ (strict or not) has a horizontal 2-category, $\mathcal{H}or\,\mathbb{A}$. The objects are those of $\mathbb{A}$, the 1-cells are the horizontal arrows of $\mathbb{A}$, and the 2-cells are the *special cells* of $\mathbb{A}$, i.e. cells of the form

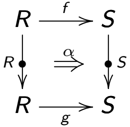
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{id}_A}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow[g]{} & B \end{array}$$

Vertical composition of 2-cells uses the canonical isomorphisms $\lambda = \rho : \mathrm{id} \bullet \mathrm{id} \longrightarrow \mathrm{id}$

$$\begin{array}{ccccccc} A & = & A & \xrightarrow{f} & B & = & B \\ & & {\scriptstyle \mathrm{id}_A}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle \mathrm{id}_B} & & \\ {\scriptstyle \mathrm{id}_A}\bullet\big\downarrow & \lambda_A^{-1} & A & \xrightarrow[g]{} & B & \lambda_B & \big\downarrow\bullet{\scriptstyle \mathrm{id}_B} \\ & & {\scriptstyle \mathrm{id}_A}\big\downarrow\bullet & \beta & \bullet\big\downarrow{\scriptstyle \mathrm{id}_B} & & \\ A & = & A & \xrightarrow[h]{} & B & = & B \end{array}$$

The 2-category $\mathcal{R}ing$

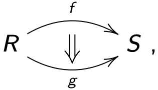
When applied to the double category $\mathbb{R}\mathrm{ing}$ we get a 2-category whose objects are rings, whose arrows are homomorphisms and whose 2-cells are linear maps of the form



Such an α is determined by its value at 1. We have $\alpha(r) = \alpha(r \cdot 1) = g(r)\alpha(1)$ and $\alpha(r) = \alpha(1 \cdot r) = \alpha(1)f(r)$ which gives the following

Definition

The 2-category of rings, $\mathcal{R}ing$, has rings as objects, homomorphisms as 1-cells and as 2-cells



elements $a \in S$ such that $af(r) = g(r)a$

Then (c) in the proposition says that f is isomorphic to g

- Let $f : A \longrightarrow B$ be a horizontal arrow in a double category $\mathbb{A}$ and $v : B \longrightarrow\!\!\!\!\!\!\bullet\;\; A$ a vertical one. We say that v is *conjoint* to f if we are given cells ψ and χ (*conjunctions*) such that

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{1_B} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \stackrel{\psi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle v} & \stackrel{\chi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[1_A]{} & A & \xrightarrow[f]{} & B
\end{array}
\;=\;
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \stackrel{\mathrm{id}_f}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B
\end{array}\,,
$$

$$
\begin{array}{ccc}
B & \xrightarrow{1_B} & B \\
{\scriptstyle v}\big\downarrow\!\bullet & \stackrel{\chi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}_B} \\
A & \xrightarrow[f]{} & B \\
{\scriptstyle \mathrm{id}_A}\big\downarrow\!\bullet & \stackrel{\psi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle v} \\
A & \xrightarrow[1_A]{} & A
\end{array}
\quad \cdot = \cdot \quad
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{\scriptstyle v}\big\downarrow\!\bullet & \stackrel{1_v}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle v} \\
A & \xrightarrow[1_A]{} & A
\end{array}\,,
$$

- In $\mathbb{R}\mathrm{ing}$, every homomorphism $f : R \longrightarrow S$ has a conjoint f^*, namely $S : S \longrightarrow\!\!\!\!\!\!\bullet\;\; R$ with left action by R given by "restriction"

$$r{\bullet}s = f(r)s$$

Rank 2

- Homomorphisms $f\colon R \longrightarrow S$ correspond to bimodules $M\colon R \longrightarrow\!\!\bullet\!\!\rightarrow S$ which are free on one generator as left S-modules
- What if M is free on 2 generators?
- Assume M free on m_1, m_2 as a left S-module. Nothing is said about the right action (as before). Then for each $r \in R$ we get unique $s_{11}, s_{12}, s_{21}, s_{22} \in S$ such that

$$\begin{array}{rcccc} m_1 r & = & s_{11} m_1 & + & s_{12} m_2 \\ m_2 r & = & s_{21} m_1 & + & s_{22} m_2 \end{array}$$

  Let's denote s_{ij} by $f_{ij}(r)$. So to each r we associate not 2 but 4 elements of S or rather a 2×2 matrix in S

Rank p

If M is free on p generators $m_1, \ldots, m_p$:

$$m_i r = \sum_{j=1}^{p} f_{ij}(r) m_j$$

Theorem

(a) Any matrix-valued homomorphism $f : R \longrightarrow Mat_p(S)$ induces an S-R-bimodule structure on $S^{(p)}$
(b) Any S-R-bimodule $M : R \longrightarrow\!\!\!\!\!\bullet\;\; S$ which is free on p generators as a left S-module is isomorphic (as on S-R-bimodule) to $S^{(p)}$ with R-action induced by a homomorphism $f : R \longrightarrow Mat_p(S)$ as in (a)
(c) The homomorphism f in (b) is unique up to conjugation by an invertible $p \times p$ matrix A in $Mat_p(S)$

Proof sketch

- Module structure on $S^{(p)}$:

  An element of $S^{(p)}$ is a row vector, i.e. a $1 \times p$ matrix $\mathbf{s} = [s_1, \ldots, s_p]$

$$s' \bullet \mathbf{s} = [s' s_1, \ldots, s' s_p] \quad \text{and} \quad \mathbf{s} \bullet r = \mathbf{s} f(r)$$

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$$s' \bullet \mathbf{s} = [s' s_1, \ldots, s' s_p] \quad \text{and} \quad \mathbf{s} \bullet r = \mathbf{s} f(r)$$

- Preservation of multiplication:

$$m_i(rr') = \sum_{k=1}^{p} f_{ik}(rr') m_k$$

and

$$\begin{aligned} (m_i r) r' &= \textstyle\sum_{j=1}^{p} f_{ij}(r) m_j r' \\ &= \textstyle\sum_{j=1}^{p} f_{ij}(r) \left(\sum_{k=1}^{p} f_{jk}(r') m_k \right) \\ &= \textstyle\sum_{k=1}^{p} \left(\sum_{j=1}^{p} f_{ij}(r) f_{jk}(r') \right) m_k \end{aligned}$$

So $f_{ik}(rr') = \sum_{j=1}^{p} f_{ij}(r) f_{jk}(r')$, i.e. we get a homomorphism

$$f : R \longrightarrow Mat_p(S)$$

into the ring of $p \times p$ matrices in S

Examples

- (Pairs of homomorphisms)
  Let $f, g : R \longrightarrow S$ be homomorphisms. Then we get a homomorphism $h : R \longrightarrow Mat_2(S)$ given by

$$h(r) = \begin{bmatrix} f(r) & 0 \\ 0 & g(r) \end{bmatrix}$$

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- (Pairs of homomorphisms)
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- (Derivations)
  Let $f : R \longrightarrow S$ be a homomorphism and d an f-derivation, i.e. an additive function $d : R \longrightarrow S$ such that

$$d(rr') = d(r)f(r') + f(r)d(r')$$

  Then we get a homomorphism $R \longrightarrow Mat_2(S)$

$$r \longmapsto \begin{bmatrix} f(r) & 0 \\ d(r) & f(r) \end{bmatrix}$$

More examples

- More generally we can consider the subring of lower triangular matrices

$$L = \left\{ \left[\begin{array}{cc} s & 0 \\ s' & s'' \end{array} \right] \middle| s, s', s'' \in S \right\}$$

Then a homomorphism $R \longrightarrow Mat_2(S)$ that factors through L corresponds to a pair of homomorphisms $f, g : R \longrightarrow S$ and a derivation d from f to g, i.e. an additive function $d : R \longrightarrow S$ such that

$$d(rr') = d(r)f(r') + g(r)d(r')$$

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$$d(rr') = d(r)f(r') + g(r)d(r')$$

- We give one more, somewhat mysterious, example to illustrate the variety of morphisms we get just in the 2×2 case. For any ring S we can construct a ring of "complex numbers" over S:

$$\mathbb{C}(S) = \left\{ \left[\begin{array}{cc} s & s' \\ -s' & s \end{array} \right] \middle| s, s' \in S \right\}$$

This is a subring of $Mat_2(S)$. A homomorphism $R \longrightarrow Mat_2(S)$ that factors through $\mathbb{C}(S)$ corresponds to two additive functions $cosd, sind : R \longrightarrow S$ with the properties

$$cosd(rr') = cosd(r)cosd(r') - sind(r)sind(r')$$

$$sind(rr') = sind(r)cod(r') + cosd(r)sind(r')$$

The graded category of rings

- Homomorphisms $f\colon R \longrightarrow Mat_p(S)$ and $g\colon S \longrightarrow Mat_q(T)$ correspond to bimodules

$$S^{(p)}\colon R \longrightarrow\!\!\bullet\!\!\longrightarrow S \quad \text{and} \quad T^{(q)}\colon S \longrightarrow\!\!\bullet\!\!\longrightarrow T,$$

and we can compose these

$$T^{(q)} \otimes_S S^{(p)} \cong T^{(pq)}$$

- This gives a composite gf

$$R \xrightarrow{f} Mat_p(S) \xrightarrow{Mat_p(g)} Mat_p Mat_q(T) \cong Mat_{pq}(T)$$

Thus we first apply f to an element $r \in R$ to get a $p \times p$ matrix in S, and then apply g to each entry separately to get a $p \times p$ block matrix of $q \times q$ matrices, and then consider this as a $(pq) \times (pq)$ matrix

Theorem

With this composition we get an $(\mathbb{N}^+, \cdot)$-graded category **Gring** *whose objects are rings and whose morphisms of degree p are homomorphisms into $p \times p$ matrices:*

$$\frac{R \xrightarrow{(p,f)} S}{f\colon R \longrightarrow Mat_p(S)}$$

The graded double category of rings

The double category $\mathbb{G}\mathrm{ring}$

- Objects rings
- Horizontal arrows $(p, f) \colon R \longrightarrow R'$
- Vertical arrows are bimodules $M \colon R \longrightarrow\!\!\!\!\!\bullet\;\; S$
- A double cell

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array}$$

is a linear map (a cell in $\mathbb{R}\mathrm{ing}$)

$$\begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle M}\big\downarrow\bullet & \overset{\phi}{\Longrightarrow} & \bullet\big\downarrow{\scriptstyle Mat_{q,p}(M')} \\ S & \xrightarrow[g]{} & Mat_q(S') \end{array}$$

where $Mat_{q,p}(M')$ is the bimodule of $q \times p$ matrices with entries in M', with the $Mat_q(S')$ action given by matrix multiplication on the left, and similarly for $Mat_p(R')$

Properties of $\mathbb{G}\mathrm{ring}$

Theorem

(1) $\mathbb{G}\mathrm{ring}$ *is a double category*

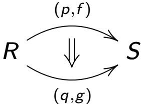
(2) Every horizontal arrow has a companion

(3) Every horizontal arrow has a conjoint

(4) The vertically full double subcategory determined by the morphisms of degree 1 *is isomorphic to* $\mathbb{R}\mathrm{ing}$

The 2-category $\mathcal{G}ring$

- The horizontal 2-category of $\mathbb{G}\mathrm{ring}$ can be described as follows:
- The objects are rings
- The morphisms are graded morphisms $(p, f)\colon R \longrightarrow S$
- A 2-cell



is given by a $q \times p$ matrix A with entries in S such that for every $r \in R$

$$Af(r) = g(r)A$$

Cauchy completeness

- If a horizontal arrow $f\colon A \longrightarrow B$ in a double category $\mathbb{A}$ has a companion f_* and a conjoint f^* then f_* is left adjoint to f^* in $\mathcal{V}ert\mathbb{A}$
- Say that B is *Cauchy complete* if every adjoint pair $v \dashv u$, $v\colon A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$, $u\colon B \longrightarrow\!\!\!\!\!\!\bullet\;\; A$ is of the form $f_* \dashv f^*$ for some $f\colon A \longrightarrow B$
- $\mathbb{A}$ is *Cauchy* if every object is Cauchy complete

Example

$\mathbb{R}\mathrm{el}$ is Cauchy

Adjoint bimodules

Recall that two bimodules $M : R \mathrel{-\!\!\bullet\!\!\!\to} S$ and $N : S \mathrel{-\!\!\bullet\!\!\!\to} R$ are adjoint, or more precisely M is left adjoint to N, if there are an S-S linear map

$$\epsilon : M \otimes_R N \longrightarrow S$$

and an R-R linear map

$$\eta : R \longrightarrow N \otimes_S M$$

such that

$$\begin{array}{ccccc} & & M \otimes_R N \otimes_S M & & \\ & \nearrow^{M\otimes_R \eta} & & \searrow^{\epsilon\otimes_S M} & \\ M \otimes_R R & \xrightarrow[\cong]{} & M & \xrightarrow[\cong]{} & S \otimes_S M \end{array}$$

and

$$\begin{array}{ccccc} & & N \otimes_S M \otimes_R N & & \\ & \nearrow^{\eta\otimes_R N} & & \searrow^{N\otimes_S \epsilon} & \\ R \otimes_R N & \xrightarrow[\cong]{} & N & \xrightarrow[\cong]{} & M \otimes_S S \end{array}$$

commute

The following theorem is well-known

Theorem

A bimodule $M : R \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\!\!\rightarrow S$ has a right adjoint if and only if it is finitely generated and projective as a left S-module

Remark

Given this theorem then, we see that S is Cauchy-complete in $\mathbb{G}\mathrm{ring}$ if and only if every finitely generated projective left S-module is free. Commutative rings with this property are of considerable interest in algebraic geometry having to do with when vector bundles are trivial. If S is a PID or a local ring then it is Cauchy. That polynomial rings are so is the content of the Quillen-Suslin theorem, which is highly non trivial

Finitely generated projective

M is finitely generated, by $m_1, \ldots, m_p$ say, if and only if the S-linear map

$$\tau : S^{(p)} \longrightarrow M$$

$\tau(s_1 \ldots s_p) = \sum_{i=1}^{p} s_i m_i$ is surjective. If M is S-projective, then τ splits, i.e. there is an S-linear map

$$\sigma : M \longrightarrow S^{(p)}$$

such that $\tau\sigma = 1_M$. In fact, M is a finitely generated and projective S-module if and only if there exist p, τ, σ such that $\tau\sigma = 1_M$

Let the components of σ be $\sigma_1, \ldots, \sigma_p : M \longrightarrow S$. Then $\tau\sigma = 1_M$ means that for every $m \in M$ we will have

$$m = \sum_{i=1}^{p} \sigma_i(m) m_i$$

i.e. the σ_i provide an S-linear choice of coordinates for m relative to the generators $m_1 \ldots m_p$. All of this is independent of R

Amplifying homomorphisms

For any r we can write

$$m_i r = \sum_{j=1}^{p} \sigma_j(m_i r) m_j$$

If we let $f_{ij}(r) = \sigma_j(m_i r)$ we get the same formula as for $\mathbb{G}\mathrm{ring}$ (on frame 15)

$$m_i r = \sum_{j=1}^{p} f_{ij}(r) m_j$$

Theorem

(1) The functions f_{ij} define a non-unital homomorphism $f : R \longrightarrow Mat_p(S)$
(2) Any such homomorphism comes from a bimodule which is finitely generated and projective as a left S-module

(Non-unital) homomorphisms $R \longrightarrow Mat_p(S)$ have already appeared in the quantum field theory literature (see e.g. Szlachanyi, K, Vecsernyes, K, Quantum symmetry and braid group statistics in G-spin models, Commun. Math. Phys. 156, 127-168 (1993)) where they are called *amplifying homomorphisms* or *amplimorphisms* for short

The double category $\mathbb{A}\mathrm{mpli}$

- Objects: rings (with 1)
- Horizontal arrows: amplimorphisms $R \longrightarrow S$,
- Vertical arrows: bimodules $M : R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$
- Cells:

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & R' \\ {\scriptstyle M}\big\downarrow\!\bullet & \overset{\phi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle M'} \\ S & \xrightarrow[(q,g)]{} & S' \end{array} \qquad \text{are cells} \qquad \begin{array}{ccc} R & \xrightarrow{f} & Mat_p(R') \\ {\scriptstyle M}\big\downarrow\!\bullet & \overset{\phi}{\Longrightarrow} & \bullet\!\big\downarrow{\scriptstyle Mat_{q,p}(M')} \\ S & \xrightarrow[g]{} & Mat_q(S') \end{array}$$

i.e. additive functions $\phi : M \longrightarrow Mat_{q,p}(M')$ such that

$$\phi(mr) = \phi(m)f(r) \qquad \phi(sm) = g(s)\phi(m)$$

Theorem

(1) $\mathbb{A}\mathrm{mpli}$ is a double category
(2) $\mathbb{A}\mathrm{mpli}$ is vertically self dual
(3) Every horizontal arrow has a companion and a conjoint
(4) $\mathbb{A}\mathrm{mpli}$ is Cauchy

Proposition

The 2-category $\mathcal{A}mpli$ of amplifying homomorphisms has unitary rings as objects, amplimorphisms $(p, f) : R \longrightarrow S$ as morphisms and as 2-cells $\phi : (p, f) \Rightarrow (q, g)$, $q \times p$ matrices A such that
(1) $Af(1) = A$
(2) *for every* $r \in R, Af(r) = g(r)A$
The identity 2-cell on (p, f) is the $p \times p$ matrix $f(1)$

Corollary

Two representations (p, f) and (q, g) of the same S-R-bimodule (finitely generated projective over S) are related as follows: There is a $q \times p$ matrix A and a $p \times q$ matrix B, both with entries in S, such that
(1) $Af(1) = A$ *and* $Af(r) = g(r)A$
(2) $Bg(1) = B$ *and* $Bg(r) = f(r)B$
(3) $AB = g(1)$ *and* $BA = f(1)$