Presentations and algebraic colimits of enriched monads for a subcategory of arities

Jason Parker
(joint with Rory Lucyshyn-Wright)

Brandon University, Manitoba

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Motivation

- Signatures and presentations for monads and theories (relative to a subcategory of arities) have been previously studied mainly in the context of *locally presentable* enriched categories; see e.g.
  - [5] G.M. Kelly and A.J. Power. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra* Vol. 89 (1993) 163-179.
  - [6] S. Lack. On the monadicity of finitary monads. *Journal of Pure and Applied Algebra* Vol. 140 (1999) 65-73.
  - [3] J. Bourke and R. Garner. Monads and theories. *Advances in Mathematics* Vol. 351 (2019) 1024-1071.

Motivation

- Kelly and Power show in [5] that if $\mathscr{C}$ is a locally finitely presentable $\mathscr{V}$-category over a locally finitely presentable closed category $\mathscr{V}$, then the forgetful functor $\mathcal{W} : \mathbf{Mnd}_f(\mathscr{C}) \to \mathbf{End}_f(\mathscr{C})$ has a left adjoint, i.e. any finitary $\mathscr{V}$-endofunctor on $\mathscr{C}$ has a free finitary $\mathscr{V}$-monad.

- They also show that the forgetful functor $\mathcal{U} : \mathbf{Mnd}_f(\mathscr{C}) \to \mathbf{Sig}_f(\mathscr{C})$ from finitary $\mathscr{V}$-monads on $\mathscr{C}$ to finitary signatures in $\mathscr{C}$ is of *descent type*. Lack then shows in [6] that $\mathcal{U} : \mathbf{Mnd}_f(\mathscr{C}) \to \mathbf{Sig}_f(\mathscr{C})$ is actually *monadic*. In particular, any finitary $\mathscr{V}$-monad on $\mathscr{C}$ then has a *presentation* in terms of abstract operations and equations.

- Bourke and Garner show in [3] that if $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is a small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally presentable $\mathscr{V}$, then $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}\text{-}\mathbf{Nerv}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ is monadic, and $\mathbf{Mnd}_{\mathscr{J}\text{-}\mathbf{Nerv}}(\mathscr{C})$ has all small (algebraic) colimits.

Motivation

- In this talk, we will discuss a very general framework for studying signatures, presentations, and algebraic colimits of enriched monads relative to a subcategory of arities, which applies even to *locally bounded* enriched categories [10] and the Borceux-Day π*-categories* [2] (which need not be locally presentable).

- Our results subsume the results of Kelly, Power, and Lack just mentioned, as well as many instances of the Bourke-Garner results just mentioned. Our results also apply in great generality to the Φ*-accessible* $\mathscr{V}$-monads studied by Lack and Rosický in
  - [7] S. Lack and J. Rosický. Notions of Lawvere theory. *Applied Categorical Structures* 19 (2011) 363-391.

Eleutheric subcategories of arities

- Fix a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$. A **subcategory of arities** $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ is a small, full, and dense sub-$\mathscr{V}$-category.

- A subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **eleutheric** if any $\mathscr{V}$-functor $F : \mathscr{J} \to \mathscr{C}$ has a left Kan extension along j, which is moreover preserved by each $\mathscr{C}(J, -) : \mathscr{C} \to \mathscr{V}$ $(J \in \mathbf{ob}\,\mathscr{J})$.

- $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is eleutheric iff j presents $\mathscr{C}$ as a free Φ-cocompletion of $\mathscr{J}$ for a class of small weights Φ.

$\mathscr{J}$-ary $\mathscr{V}$-monads

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is a subcategory of arities, then a $\mathscr{V}$-endofunctor $H : \mathscr{C} \to \mathscr{C}$ is $\mathscr{J}$**-ary** if it preserves left Kan extensions along j. We let $\mathbf{End}_{\mathscr{J}}(\mathscr{C})$ be the category of $\mathscr{J}$-ary $\mathscr{V}$-endofunctors on $\mathscr{C}$ and $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ the category of $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{C}$.

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ presents $\mathscr{C}$ as a free Φ-cocompletion of $\mathscr{J}$ (and hence is eleutheric), then $H : \mathscr{C} \to \mathscr{C}$ is $\mathscr{J}$-ary iff H preserves Φ-colimits.

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is eleutheric, then $\mathscr{V}\text{-}\mathbf{CAT}(\mathscr{J}, \mathscr{C}) \simeq \mathbf{End}_{\mathscr{J}}(\mathscr{C})$ via precomposition with j and left Kan extension along j.

- If $\mathscr{C} = \mathscr{V}$ and $j : \mathscr{J} \hookrightarrow \mathscr{V}$ is an eleutheric *system* of arities (i.e. $\mathscr{J}$ contains I and is closed under $\otimes$), then $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{V}) \simeq \mathbf{Th}_{\mathscr{J}}$, the category of $\mathscr{J}$**-theories** [8].

Examples of eleutheric subcategories of arities

- The subcategory of arities $j : \mathscr{C}_\alpha \hookrightarrow \mathscr{C}$ of α-presentable objects in a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable closed category $\mathscr{V}$. The $\mathscr{J}$-ary $\mathscr{V}$-endofunctors preserve conical α-filtered colimits, and the $\mathscr{J}$-ary $\mathscr{V}$-monads correspond to the enriched Lawvere theories of Nishizawa-Power [12].

- The subcategory of arities $j : \mathscr{C}_\Phi \hookrightarrow \mathscr{C}$ consisting of suitable Φ-presentable objects in a locally Φ-presentable $\mathscr{V}$-category for a suitable class of weights Φ [7]. The $\mathscr{J}$-ary $\mathscr{V}$-endofunctors preserve Φ-flat colimits, and the $\mathscr{J}$-ary (i.e. Φ-**accessible**) $\mathscr{V}$-monads correspond to the Lawvere Φ-theories of Lack-Rosický [7].

- In particular, the subcategory of arities $j : \mathscr{C}_{\mathbb{D}} \hookrightarrow \mathscr{C}$ of $\mathbb{D}$-presentable objects in a locally $\mathbb{D}$-presentable $\mathscr{V}$-category over a locally $\mathbb{D}$-presentable closed category $\mathscr{V}$ for a sound doctrine $\mathbb{D}$ [1], where the $\mathscr{J}$-ary $\mathscr{V}$-endofunctors preserve conical $\mathbb{D}$-filtered colimits.

Examples of eleutheric subcategories of arities

- The subcategory of arities $j : \{I\} \hookrightarrow \mathscr{V}$ consisting of the unit object. The $\mathscr{J}$-ary $\mathscr{V}$-endofunctors have the form $X \otimes (-) : \mathscr{V} \to \mathscr{V}$ ($X \in \mathbf{ob}\mathscr{V}$), and the $\mathscr{J}$-ary $\mathscr{V}$-monads correspond to monoids in $\mathscr{V}$.

- The subcategory of arities $\mathbf{y}_{\mathscr{A}} : \mathscr{A} \to [\mathscr{A}^{\mathbf{op}}, \mathscr{V}]$ consisting of the representables for a small $\mathscr{V}$-category $\mathscr{A}$. The $\mathbf{y}_{\mathscr{A}}$-ary $\mathscr{V}$-endofunctors preserve small colimits, and the $\mathbf{y}_{\mathscr{A}}$-ary $\mathscr{V}$-monads correspond to identity-on-objects $\mathscr{V}$-functors with domain $\mathscr{A}^{\mathbf{op}}$.

- The subcategory of arities $j : \mathbb{N}_{\mathscr{V}} \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the unit object in any π-category $\mathscr{V}$ [2]. The $\mathscr{J}$-ary $\mathscr{V}$-endofunctors preserve $\mathbb{N}_{\mathscr{V}}$-flat colimits (incl. sifted colimits), and the $\mathscr{J}$-ary $\mathscr{V}$-monads correspond to the enriched finite power theories of Borceux-Day [2].

Factegories

- We will now discuss the other assumption(s) that we will need to impose on our subcategories of arities.
- $\mathscr{V}$ is a **closed factegory** if $\mathscr{V}$ is equipped with an enriched factorization system $(\mathscr{E}, \mathscr{M})$ [9].
- If $\mathscr{V}$ is a closed factegory, then a $\mathscr{V}$-**factegory** is a $\mathscr{V}$-category $\mathscr{C}$ equipped with an enriched factorization system $(\mathscr{E}_\mathscr{C}, \mathscr{M}_\mathscr{C})$ that **is compatible with** $(\mathscr{E}, \mathscr{M})$, i.e. each $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V}$ $(C \in \mathbf{ob}\mathscr{C})$ preserves the right class.
- The $\mathscr{V}$-factegory $\mathscr{C}$ is **cocomplete** if $\mathscr{C}$ is cocomplete and has arbitrary cointersections of $\mathscr{E}_\mathscr{C}$-morphisms.

Boundedness

- Let $\mathscr{C}$ and $\mathscr{D}$ be $\mathscr{V}$-factegories with conical α-filtered colimits. A $\mathscr{V}$-functor $F : \mathscr{C} \to \mathscr{D}$ is α-**bounded** if for any α-filtered diagram $D : \mathscr{A} \to \mathscr{C}_0$ with colimit **colim** D and any $\mathscr{M}$-cocone $m = (m_A : DA \to C)_A$, if **colim** $D \xrightarrow{\overline{m}} C$ lies in $\mathscr{E}$, then **colim** $FD \xrightarrow{\overline{Fm}} FC$ lies in $\mathscr{E}$. (In Kelly's terminology [4], we also say that F **preserves the $\mathscr{E}$-tightness of α-filtered $\mathscr{M}$-cocones**.)

- We say that $F : \mathscr{C} \to \mathscr{D}$ is **bounded** if F is α-bounded for some α. If $C \in \mathbf{ob}\mathscr{C}$, then C is α-**bounded** if $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V}$ is α-bounded.

- If each of $(\mathscr{E}, \mathscr{M}), (\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}}), (\mathscr{E}_{\mathscr{D}}, \mathscr{M}_{\mathscr{D}})$ is just $(\mathbf{Iso}, \mathbf{All})$, then $F : \mathscr{C} \to \mathscr{D}$ is α-bounded iff F preserves conical α-filtered colimits.

Bounded subcategories of arities

If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is a subcategory of arities in a $\mathscr{V}$-factegory $\mathscr{C}$, then $\mathscr{J}$ is **(α-)bounded** if every $J \in \mathbf{ob}\,\mathscr{J}$ is (α-)bounded.

Proposition

If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is an α-bounded subcategory of arities in a cocomplete $\mathscr{V}$-factegory $\mathscr{C}$ and $H : \mathscr{C} \to \mathscr{C}$ is a left Kan extension along j, then H is α-bounded. In particular, any $\mathscr{J}$-ary $\mathscr{V}$-endofunctor $H : \mathscr{C} \to \mathscr{C}$ is α-bounded.

Blanket assumptions

All of our results hold for any subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ satisfying the following assumptions:

- $\mathscr{V}$ is complete, cocomplete, and a closed factegory.
- $\mathscr{C}$ is a cocomplete $\mathscr{V}$-factegory that is cotensored, and $(\mathscr{E}, \mathscr{M})$ is **proper** or $\mathscr{C}$ is $\mathscr{E}$-cowellpowered.
- $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is bounded and eleutheric.

Aside from the Lack-Rosický example, **all** of the previous examples satisfy these blanket assumptions. If $\mathscr{V}$ is *locally bounded* and $\mathscr{E}$-cowellpowered, then the Lack-Rosický example **does** satisfy these blanket assumptions (specifically, the canonical subcategory of arities is **bounded**).

Algebraically free $\mathscr{J}$-ary $\mathscr{V}$-monads

- A $\mathscr{V}$-monad $\mathbb{T}$ on $\mathscr{C}$ is an **algebraically free $\mathscr{V}$-monad** on a $\mathscr{V}$-endofunctor $H : \mathscr{C} \to \mathscr{C}$ if $U^{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Alg} \to \mathscr{C}$ is isomorphic to $U^H : H\text{-}\mathbf{Alg} \to \mathscr{C}$ in $\mathscr{V}\text{-}\mathbf{CAT}/\mathscr{C}$ (cf. [4]).

- Kelly showed [4] that $H : \mathscr{C} \to \mathscr{C}$ has an algebraically free $\mathscr{V}$-monad $\mathbb{T}_H$ iff $U^H : H\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, in which case $\mathbb{T}_H$ is the induced $\mathscr{V}$-monad.

Theorem

If $H : \mathscr{C} \to \mathscr{C}$ is $\mathscr{J}$-ary, then $U^H : H\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, and the resulting algebraically free $\mathscr{V}$-monad $\mathbb{T}_H$ on H is $\mathscr{J}$-ary. So $\mathcal{W} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{End}_{\mathscr{J}}(\mathscr{C})$ is monadic.

$\mathscr{J}$-signatures and their algebras

- A $\mathscr{J}$-**signature** in $\mathscr{C}$ is a functor $\Sigma : \mathbf{ob}\,\mathscr{J} \to \mathscr{C}_0$. The category $\mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-signatures is then the functor category $\mathbf{CAT}\,(\mathbf{ob}\,\mathscr{J}, \mathscr{C}_0)$.

- A Σ-**algebra** in $\mathscr{C}$ is an object $C \in \mathbf{ob}\mathscr{C}$ equipped with morphisms $\Sigma J \to [\mathscr{C}(J, C), C]$ $(J \in \mathbf{ob}\,\mathscr{J})$, yielding a $\mathscr{V}$-category $U^{\Sigma} : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ over $\mathscr{C}$.

- We have a canonical forgetful functor $\mathcal{V} : \mathbf{End}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$, which we have shown is monadic. The free $\mathscr{J}$-ary $\mathscr{V}$-endofunctor on a $\mathscr{J}$-signature Σ is the *polynomial* $\mathscr{V}$-endofunctor $H_{\Sigma} : \mathscr{C} \to \mathscr{C}$, given by

$$(C \in \mathbf{ob}\mathscr{C}) \qquad C \mapsto \coprod_{J \in \mathscr{J}} \mathscr{C}(J, C) \otimes \Sigma J.$$

Free $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{J}$-signatures

Theorem

Every $\mathscr{J}$-signature Σ has a free $\mathscr{J}$-ary $\mathscr{V}$-monad $\mathbb{T}_\Sigma$ with $\mathbb{T}_\Sigma$-$\mathbf{Alg} \cong \Sigma$-$\mathbf{Alg}$ in $\mathscr{V}$-$\mathbf{CAT}/\mathscr{C}$. So $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ has a left adjoint.

Using Lack's result [6, Theorem 2], we have also shown:

Theorem

$\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ is monadic.

Algebraic colimits of $\mathscr{V}$-monads

- An **algebraic colimit** of a small diagram of $\mathscr{V}$-monads $\mathbb{M} : \mathcal{K} \to \mathbf{Mnd}(\mathscr{C})$ is a colimit $\mathbb{T}$ that is sent to a limit by the (fully faithful) semantics functor $\mathbf{Alg} : \mathbf{Mnd}(\mathscr{C})^{\mathbf{op}} \to \mathscr{V}\text{-}\mathbf{CAT}/\mathscr{C}$ (cf. [4]).

- Let $U^{\mathbb{M}} : \mathbb{M}\text{-}\mathbf{Alg} \to \mathscr{C}$ be the limit of

$$\mathcal{K}^{\mathbf{op}} \xrightarrow{\mathbb{M}^{\mathbf{op}}} \mathbf{Mnd}(\mathscr{C})^{\mathbf{op}} \xrightarrow{\mathbf{Alg}} \mathscr{V}\text{-}\mathbf{CAT}/\mathscr{C}.$$

  Kelly showed [4] that $\mathbb{M}$ has an algebraic colimit $\mathbb{T}_{\mathbb{M}}$ iff $U^{\mathbb{M}} : \mathbb{M}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, in which case $\mathbb{T}_{\mathbb{M}}$ is the induced $\mathscr{V}$-monad.

Algebraic colimits of $\mathscr{J}$-ary $\mathscr{V}$-monads

Theorem

Let $\mathbb{M} : \mathcal{K} \to \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ be a small diagram. Then $U^{\mathbb{M}} : \mathbb{M}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, and the resulting algebraic colimit $\mathbb{T}_{\mathbb{M}}$ is $\mathscr{J}$-ary. So $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ has small algebraic colimits.

Presentations of $\mathscr{J}$-ary $\mathscr{V}$-monads

- A $\mathscr{J}$-**presentation** is a parallel pair of $\mathscr{J}$-signature morphisms $\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma$. Γ should be thought of as the **signature of equations**.

- A $\mathscr{J}$-presentation $(\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma)$ **presents** a $\mathscr{J}$-ary $\mathscr{V}$-monad $\mathbb{T}$ if $\mathbb{T}$ is an algebraic coequalizer of the induced parallel pair $\bar{\alpha}, \bar{\beta} : \mathbb{T}_\Gamma \rightrightarrows \mathbb{T}_\Sigma$, i.e. there is a coequalizer $\mathbb{T}_\Gamma \rightrightarrows \mathbb{T}_\Sigma \twoheadrightarrow \mathbb{T}$ in $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ preserved by the semantics functor $\mathbf{Alg} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to (\mathscr{V}\text{-}\mathbf{CAT}/\mathscr{C})^{\mathbf{op}}$.

Theorem

Every $\mathscr{J}$-presentation P presents a $\mathscr{J}$-ary $\mathscr{V}$-monad $\mathbb{T}_P$, and every $\mathscr{J}$-ary $\mathscr{V}$-monad has a (canonical) $\mathscr{J}$-presentation.

Algebras for $\mathscr{J}$-presentations

- Let $P = (\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma)$ be a $\mathscr{J}$-presentation. If A is a Σ-algebra, then for every $J \in \mathbf{ob}\,\mathscr{J}$ there is a canonical morphism $\gamma^A_J : T_\Sigma J \to [\mathscr{C}(J, A), A]$.

- A P-**algebra** is a Σ-algebra A for which

$$\gamma^A_J \circ \alpha_J = \gamma^A_J \circ \beta_J : \Gamma J \rightrightarrows T_\Sigma J \to [\mathscr{C}(J, A), A]$$

  for all $J \in \mathbf{ob}\,\mathscr{J}$. We then have the full sub-$\mathscr{V}$-category $P\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$ spanned by the P-algebras.

Theorem

Let $P = (\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma)$ be a $\mathscr{J}$-presentation, and let $\mathbb{T}_P$ be the $\mathscr{J}$-ary $\mathscr{V}$-monad presented by P. Then $\mathbb{T}_P\text{-}\mathbf{Alg} \cong P\text{-}\mathbf{Alg}$ in $\mathscr{V}\text{-}\mathbf{CAT}/\mathscr{C}$.

Locally bounded $\mathscr{V}$-categories

Theorem

Let $\mathscr{C}$ be a locally bounded $\mathscr{V}$-category over a locally bounded closed category $\mathscr{V}$. Then any subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is bounded.

- Thus, all of our results hold for any eleutheric subcategory of arities in any locally bounded $\mathscr{V}$-category over a locally bounded closed category $\mathscr{V}$.
- In particular, if $\mathscr{V}$ is locally bounded and $\mathscr{E}$-cowellpowered and Φ is a class of weights studied by Lack-Rosický [7], then any locally Φ-presentable $\mathscr{V}$-category $\mathscr{C}$ is locally bounded, and hence its canonical subcategory of arities $j : \mathscr{C}_\Phi \hookrightarrow \mathscr{C}$ is bounded (and eleutheric).

In summary...

- We have developed a very general framework for studying presentations and algebraic colimits for enriched monads relative to a subcategory of arities $\mathscr{J}$, which includes locally bounded enriched categories and Borceux-Day π-categories.

- Specifically, we have seen that free $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{J}$-ary $\mathscr{V}$-endofunctors and $\mathscr{J}$-signatures, algebraic colimits of $\mathscr{J}$-ary $\mathscr{V}$-monads, and presentations of $\mathscr{J}$-ary $\mathscr{V}$-monads can be obtained for any bounded and eleutheric subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ satisfying mild assumptions.

- In particular, our results apply to any eleutheric subcategory of arities in any locally bounded $\mathscr{V}$-category over a locally bounded closed category $\mathscr{V}$.

Some next steps...

- Preprint should (hopefully!) be out in the next few weeks.
- Develop (more) specific applications of these results, especially in the (new) context of eleutheric subcategories of arities in locally bounded $\mathscr{V}$-categories.
- Develop analogous results for $\mathscr{J}$-**theories** (which may require even weaker assumptions on $\mathscr{J}$).

Thank you!

E-mail: parkerj@brandonu.ca
Website: www.jasonparkermath.com

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