Orbispace Mapping Objects: Three Approaches, Two Results

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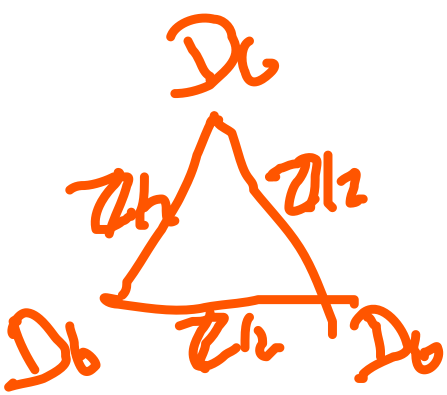
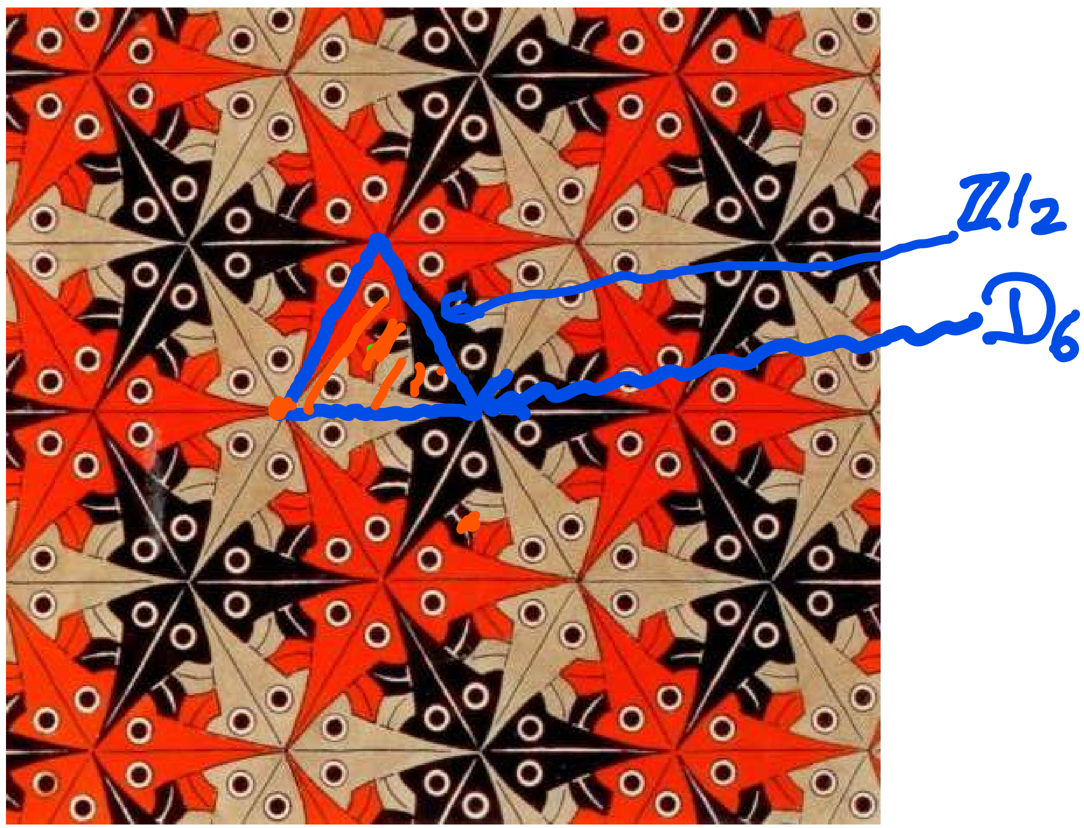
Outline

Orbispaces

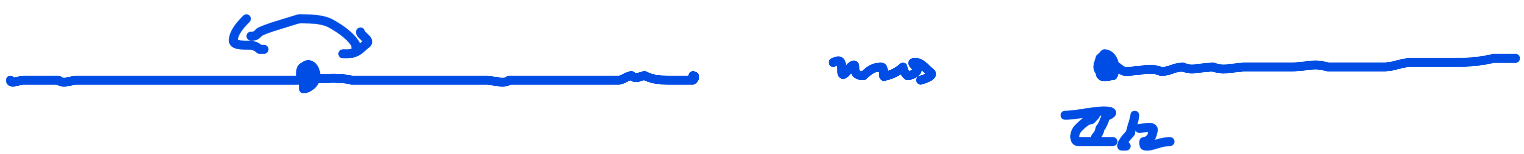
- An orbispace is a space that is obtained from another space by quotienting out (finite) symmetry, but you want to keep the info about the symmetry.
- Symmetry is modeled locally by actions of finite groups.





Example 1: Reflections

- The line with reflection symmetry about the origin modeled through an action by $\mathbb{Z}/2$. Here, the underlying space of the orbifold is a half-line with a special end-point.



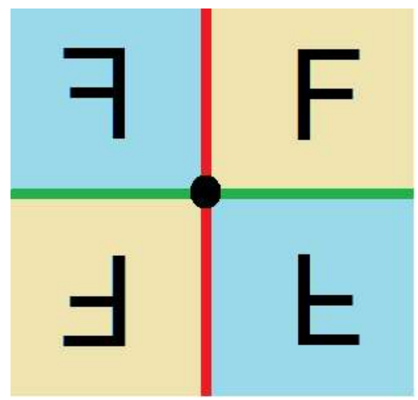
- The plane with reflection symmetry about the y-axis, modeled through an action by $\mathbb{Z}/2$. Here, the underlying space of the orbifold is a half-plane with a special boundary line.

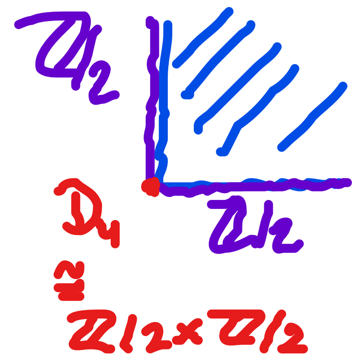
- We will call the boundaries of a space formed through this type of action, *silvered boundaries*.

Example 2: Dihedral Groups

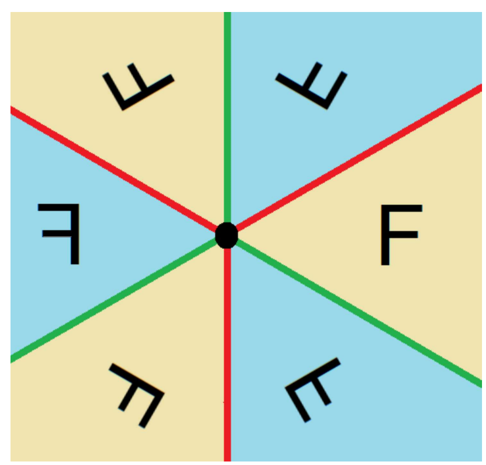
We can combine reflections to form the dihedral groups D_{2n}.

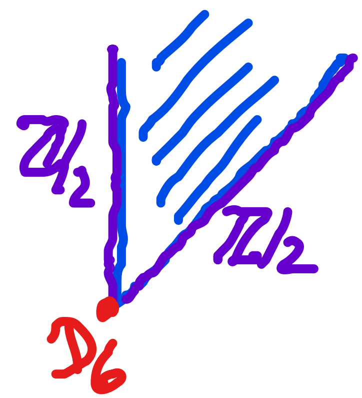
- A corner of order 2 (group D_4):





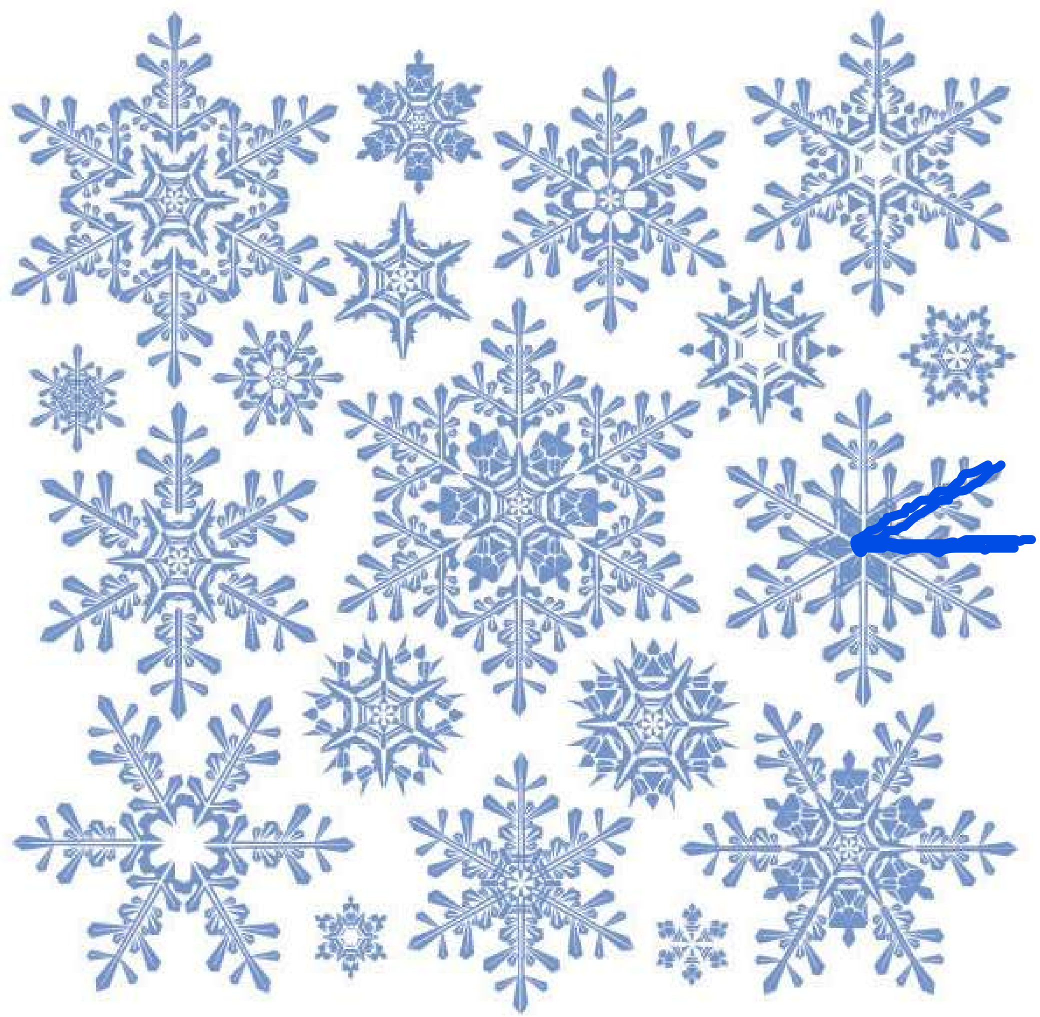
- A corner of order 3 (group D_6):





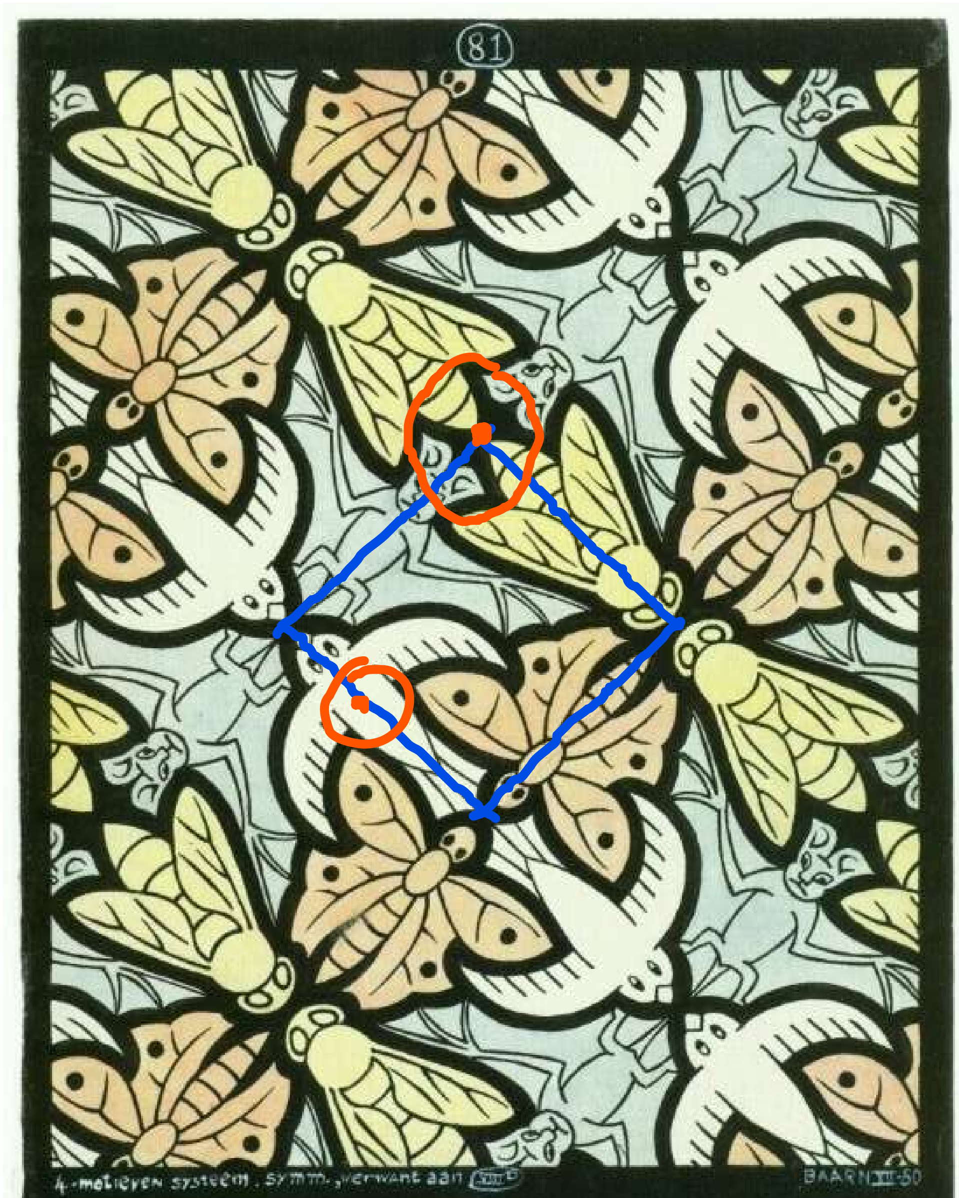
A Corner of Order 6

Corners of order 6 (group D_{12}):

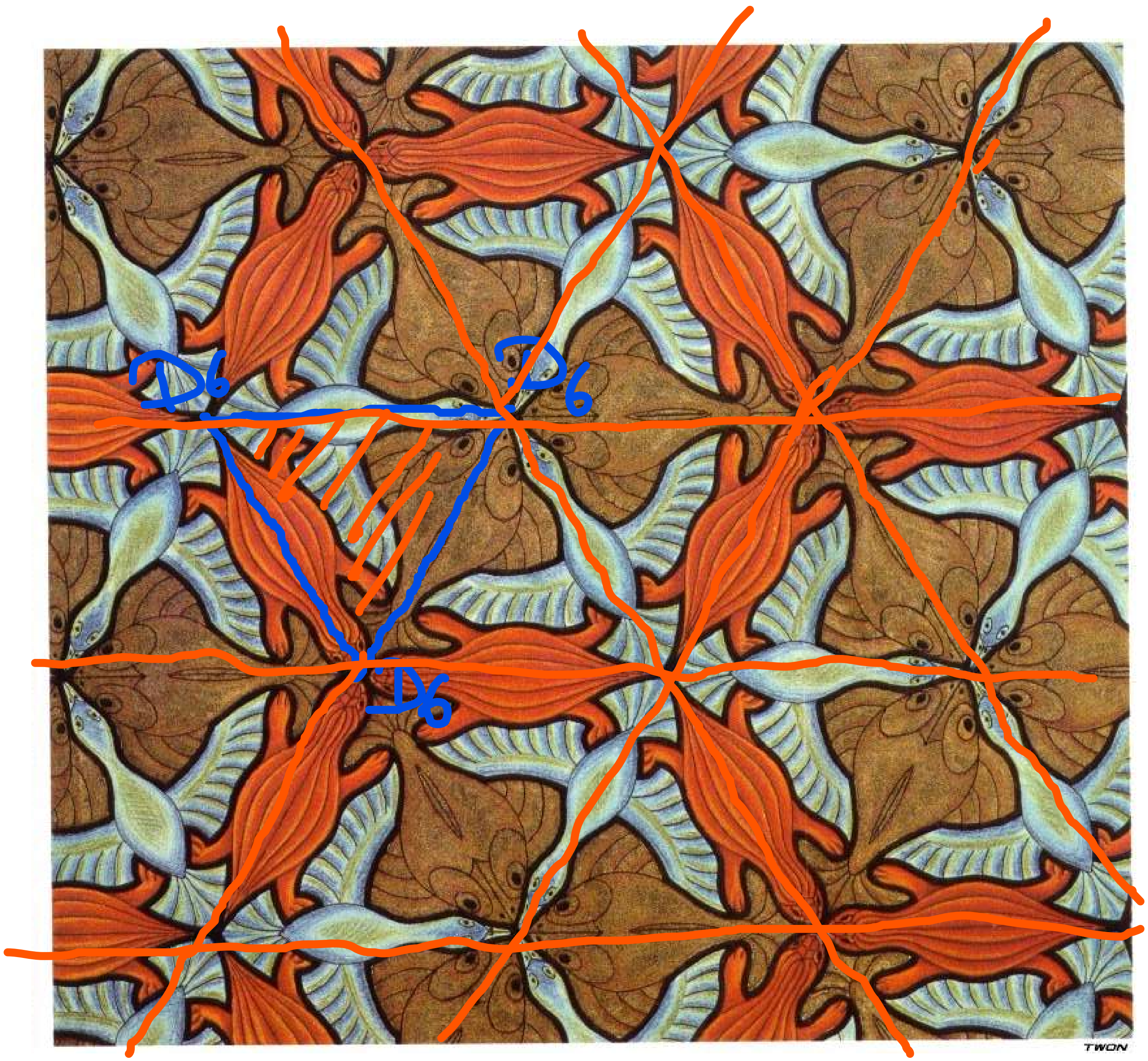


A Rectangular Billiard

<u>Note</u>: the tesselation symmetry group is infinite, but the isotropy groups are finite; local information can be given in terms of finite groups.

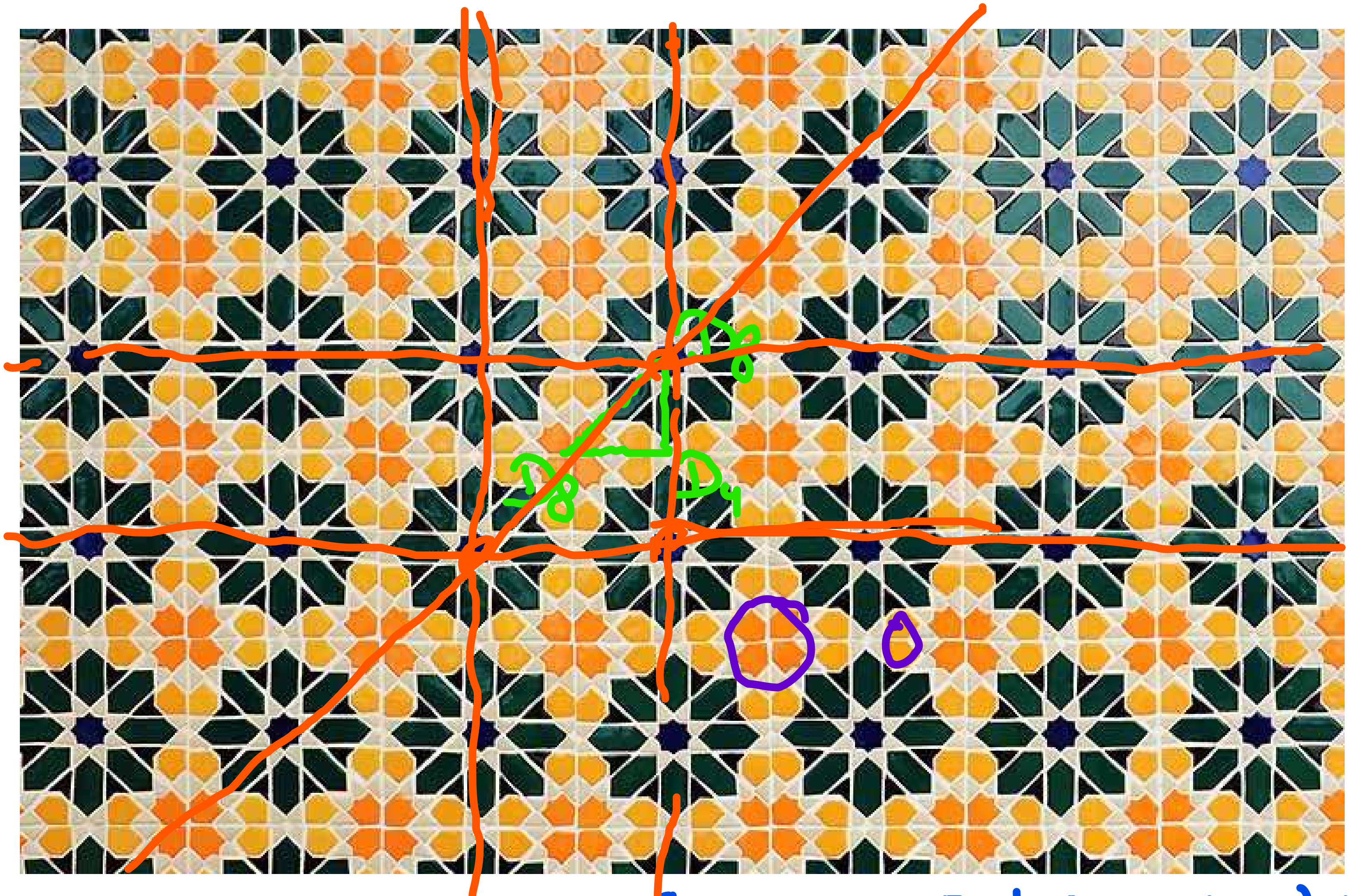


A Triangular Billiard



60°-60°-60°
triangular
billiard
D_6-D_6-D_6

Another Triangular Billiard

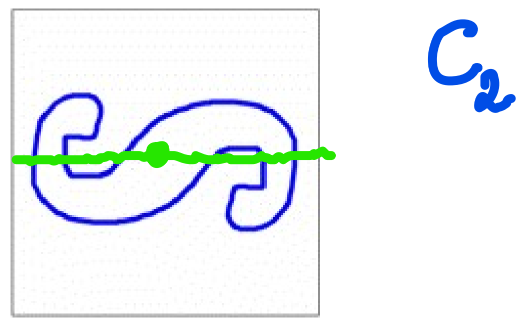


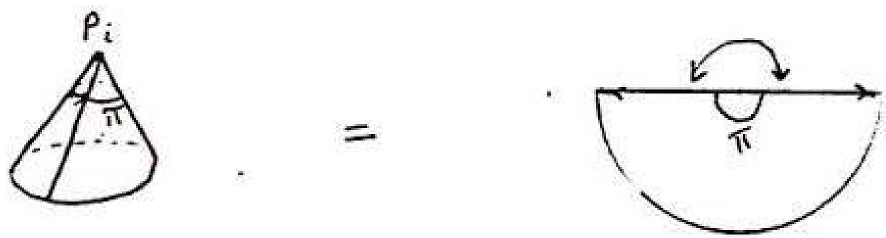
90°- 45°- 45° triangular billiard
$D_4 - D_8 - D_8$

Example 3: Actions by Rotation

The cyclic group of order *n* acts on the disc by rotation to define an orbifold with underlying space a cone of order *n*.

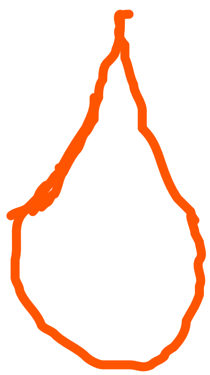
- An order 2 cone:

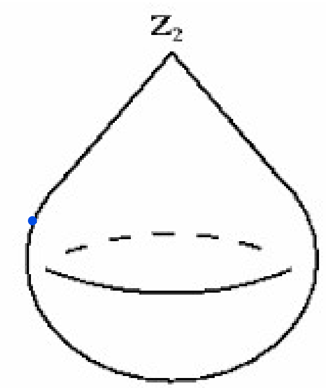




Order 3 Cones

The teardrop orbifold





A Teardrop Orbifold

We can glue together orbifolds to make more complicated ones..

What is the best way to represent these?

We need spaces and groups.

Topological Groupoids

A **topological groupoid** $\mathcal{G}$, is a groupoid internal to the category of topological spaces and continuous maps:

- a small groupoid where both the set of objects $\mathcal{G}_0$ and the set of arrows $\mathcal{G}_1$ are equipped with a topology and all structure maps are continuous:

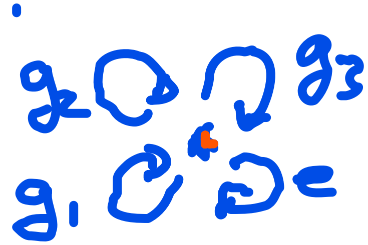
$$\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{\mu} \mathcal{G}_1 \xrightarrow{\text{inv}} \mathcal{G}_1 \underset{t}{\overset{s}{\underset{\xleftarrow{u}}{\rightrightarrows}}} \mathcal{G}_0$$

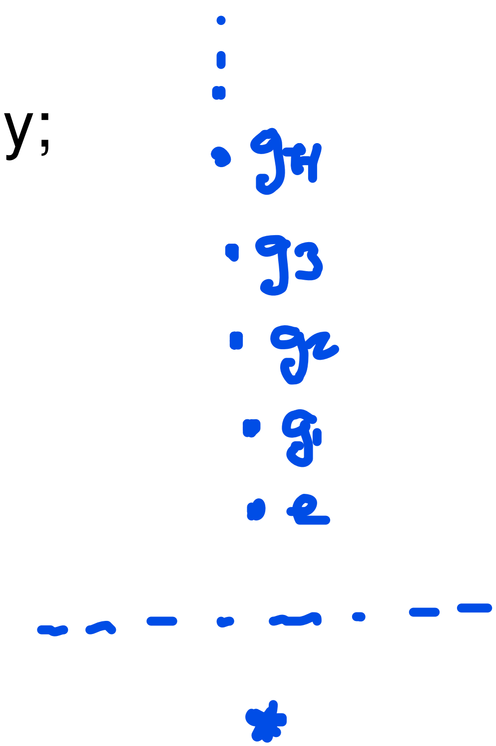
A point with a group

Given a group G, we define BG to be the groupoid with

- $(BG)_0 = \{*\}$;
- $(BG)_1 = G$, equipped with the discrete topology;
- $s(g) = t(g) = *$, for all $g \in G$;
- $\mu(g_1, g_2) = g_1 g_2$, multiplication in G;
- $u(*) = e$, the unit element of G.

think of these as:





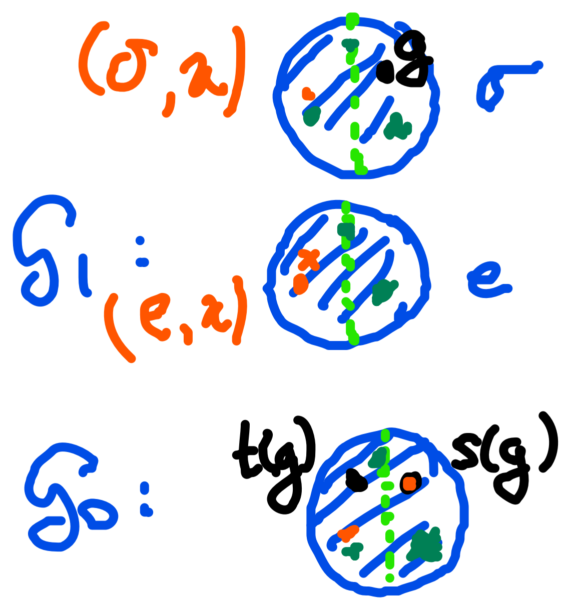
A Disc with $\mathbb{Z}/2$ Reflection Action

$$\mu((\sigma,\sigma x),(\sigma,x)) = (\sigma\sigma, x) = (e,x)$$

The disc D with $\mathbb{Z}/2$ action can be modeled as a groupoid:

- $\mathcal{G}_0 = D$;
- $\mathcal{G}_1 = \{e\} \times D \cup \{\sigma\} \times D$;
- $s(e,x) = x = s(\sigma,x)$;
- $t(e,x) = x$ and $t(\sigma,x) = \sigma(x)$.

$$t(g) \overset{g}{\longleftarrow} s(g)$$



Translation/Action Groupoids

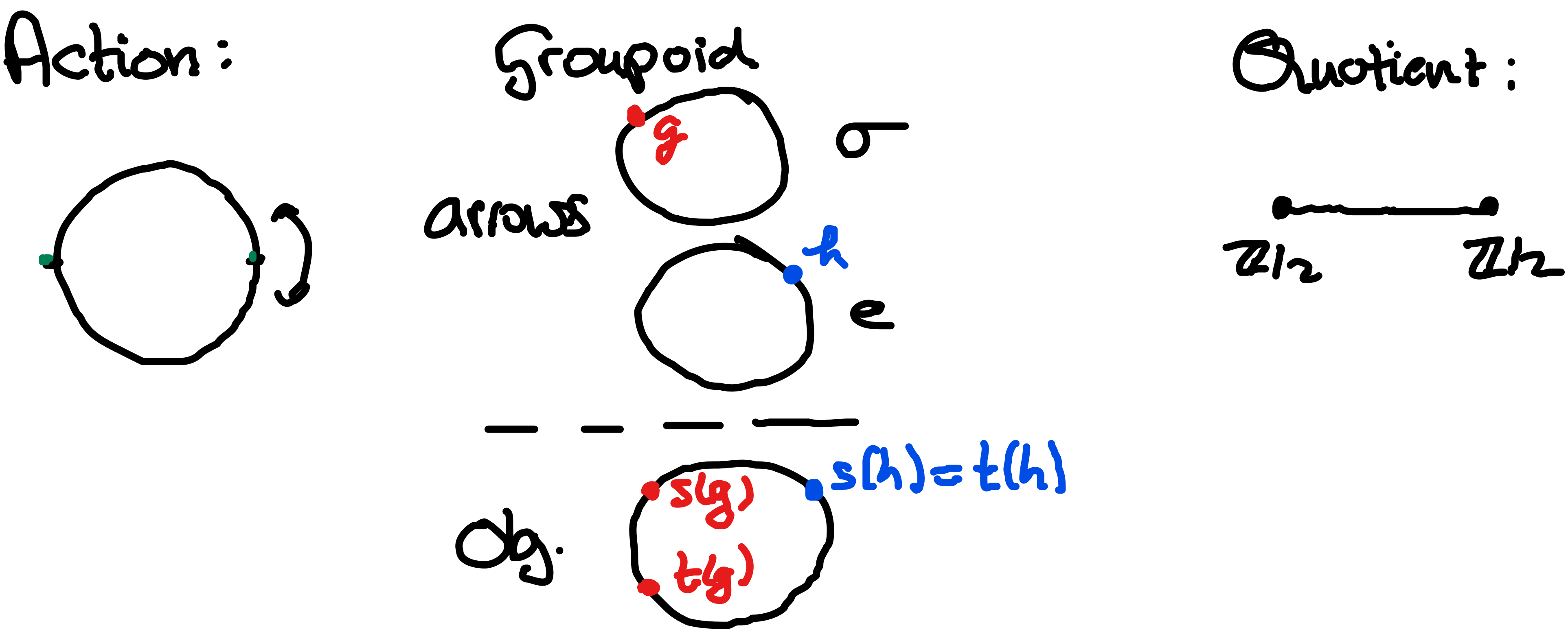
Given a discrete group G acting on a space X, $G \times X \xrightarrow{\cdot} X$, the translation groupoid $G \ltimes X$ is defined as follows:

- $(G \ltimes X)_0 = X$ and $(G \ltimes X)_1 = G \times X$;
- $s(g, x) = x$ and $t(g, x) = g \cdot x$;
- $\mu((g, h \cdot x), (h, x)) = (gh, x)$ and $\mathrm{inv}(g, x) = (g^{-1}, g \cdot x)$.

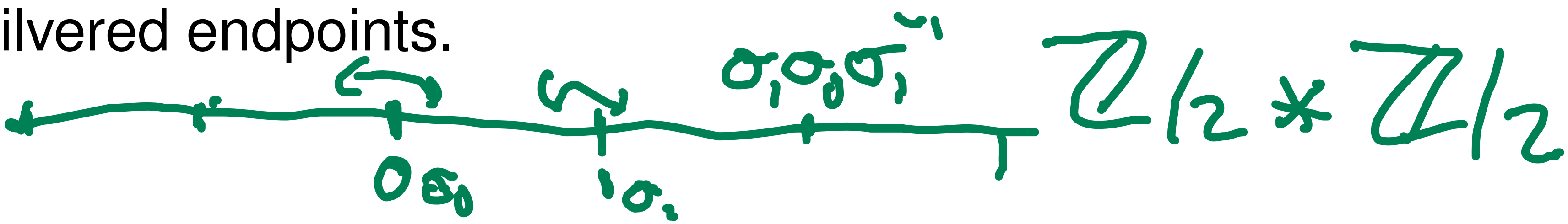
$\cdot\, x \in X$

$$\begin{array}{c} G \times X \\ \downdownarrows \\ \hline X \end{array}$$

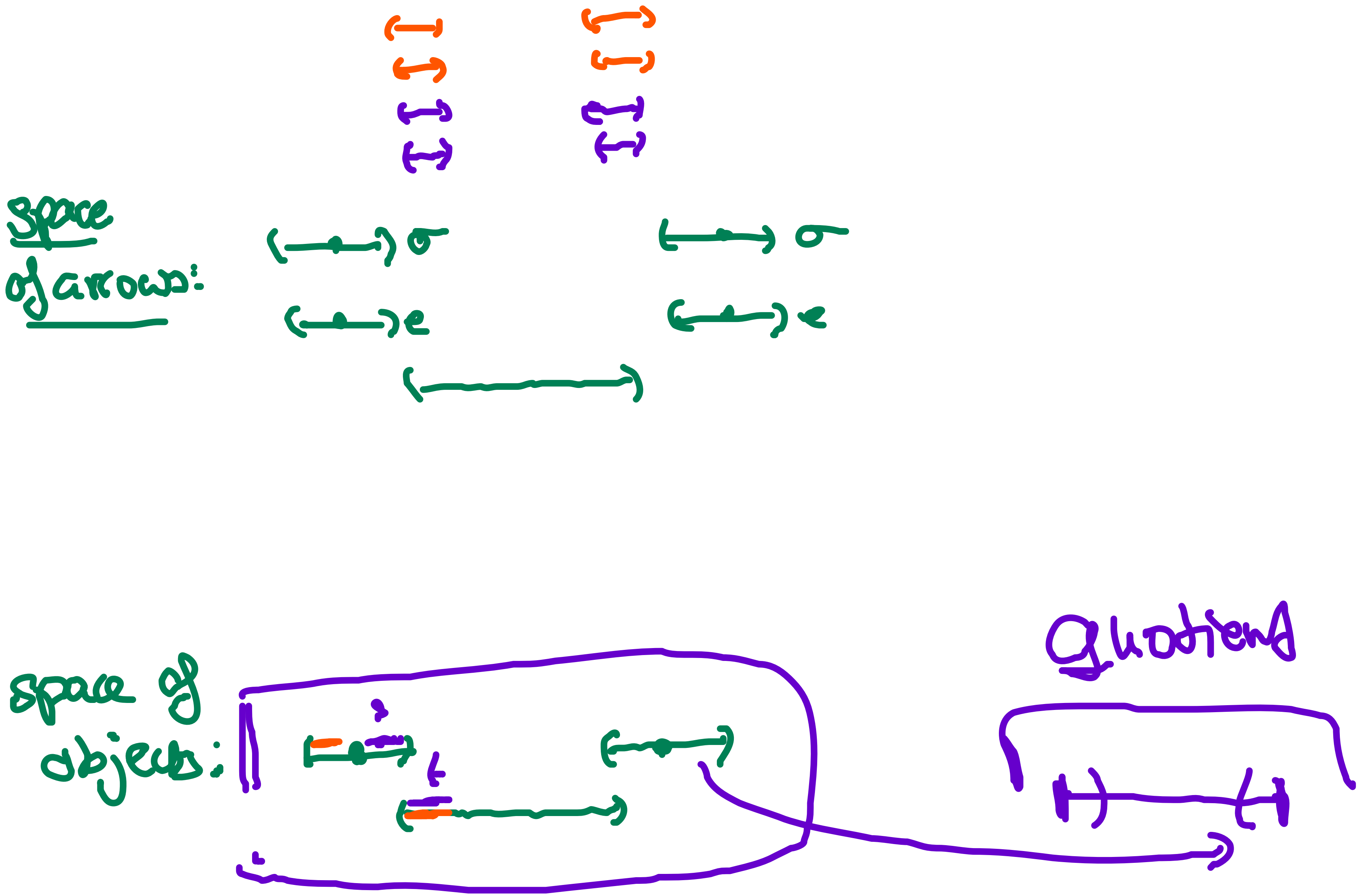
S^1 with a $\mathbb{Z}/2$-Action



This orbifold is also called the ‘silvered interval’, or the interval with silvered endpoints.



Another presentation of the silvered interval



Orbispaces as Groupoids

local homeomorphism

Orbispaces are proper étale groupoids internal to the category of Hausdorff spaces: manifolds – locally Euclidean.

- $\mathcal{G}_1 \xrightarrow{(s,t)} \mathcal{G}_0 \times \mathcal{G}_0$ is proper (closed with compact fibers);
- the source map $\mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0$ is étale (and hence all structure maps are étale).

- Satake '56, 57
- Thurston

Kawasaki

Chen Ruan ...

Groupoid Homomorphisms

Definition

A morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ between topological groupoids is a continuous functor; i.e., a pair of continuous maps

$$\varphi_0 : \mathcal{G}_0 \to \mathcal{H}_0 \text{ and } \varphi_1 : \mathcal{G}_1 \to \mathcal{H}_1$$

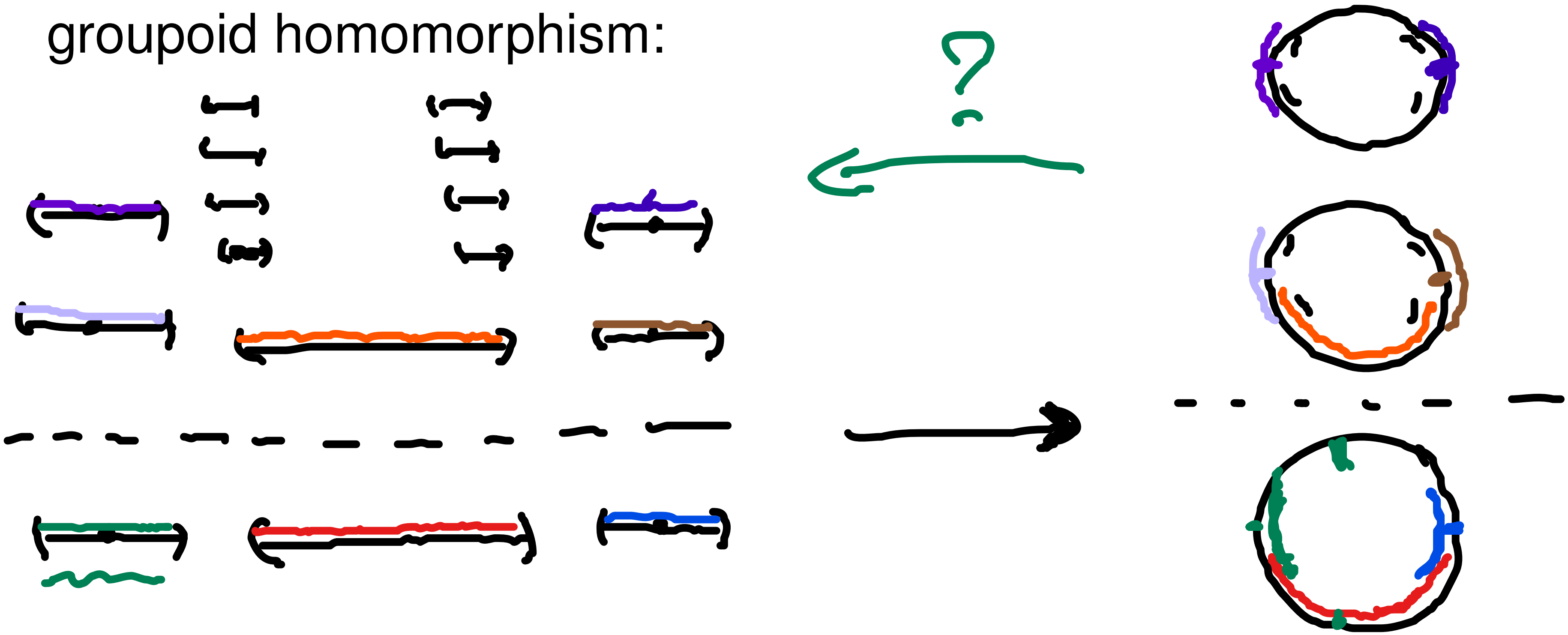
that makes the usual diagrams commute:

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "\mu"] \arrow[d, "(\varphi_1{,}\varphi_1)"'] & \mathcal{G}_1 \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] \arrow[d, "\varphi_1"'] & \mathcal{G}_0 \arrow[l, "u" description] \arrow[d, "\varphi_0"] \\
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r, "\mu"'] & \mathcal{H}_1 \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & \mathcal{H}_0 \arrow[l, "u" description]
\end{tikzcd}$$

However, this isn't all...

Multiple representations for the same orbispace

We have seen two representations for the silvered interval. There is a groupoid homomorphism:



However, this is not an isomorphism, or even an equivalence. It is an example of an essential equivalence.

Essential Equivalences

- A morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is *essentially surjective* and *fully faithful*.
- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_0$ in

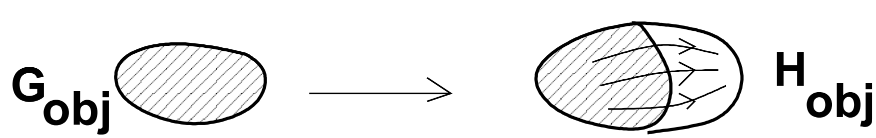
(p, g) $\quad$ g

$$\begin{tikzcd}
\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[r, "t"] \arrow[d, "s"] & \mathcal{H}_0 \\
\mathcal{G}_0 \arrow[r, "\varphi_0"'] & \mathcal{H}_0 \arrow[ur, bend left] &
\end{tikzcd}$$

x

p $\quad$ y $\quad$ $\varphi_0(p)$

is an **open surjection**.



f may not be onto the objects of $\mathcal{H}$, but every object in $\mathcal{H}_0$ is isomorphic to an object in the image of $\mathcal{G}_0$.

Essential Equivalences

A morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ is **fully faithful** if the following square is a pullback:

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "\varphi_1"] \arrow[d, "{(s,t)}"'] & \mathcal{H}_1 \arrow[d, "{(s,t)}"] \\ \mathcal{G}_0 \times \mathcal{G}_0 \arrow[r, "\varphi_0\times\varphi_0"'] & \mathcal{H}_0 \times \mathcal{H}_0 \end{tikzcd}$$