Model bicategories & their homotopy bicategories

Martin Szyld (joint work with
Maria Emilia Descotte & Eduardo Dubuc)

ATCAT, Dalhousie University

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Why Model bi categories ??

- $\mathcal{C}$ bicategory, three families of arrows

  $\mathcal{F}$ fibrations $\cdot \xrightarrow{p} \twoheadrightarrow \cdot$ co$\mathcal{F}$ cofibrations $\cdot \overset{i}{\hookrightarrow} \cdot$

  $\mathcal{W}$ weak equivalences $\cdot \xrightarrow{\sim} \cdot$

  Satisfying (autodual) ~~axioms~~ that allow to:

- define a homotopy $f \overset{\sim}{\Rightarrow} g$ between arrows

- compute its homotopy bicategory in which the 2-cells are given by the homotopies (bicateg.)

  arrows by this relation. Ho(C)

  develop and prove that this yields fibrant replacement $\mathcal{C} \longrightarrow Ho(\mathcal{C})$ localiz. at W.

Why model bicategories?

[Q] → Top

→ shape theory $2\text{-Pro}(\mathcal{C}) \rightsquigarrow Ho(2\text{Pro}(\mathcal{C}))$

→ Alg/c.t. $(\infty,2)$-categ.

[H] ⇝ question on model 2-categories

Definitions in [DPhD], [BPhD]

Related Notions

- 2-factorization systems [DV]
- Bicateg. fibration structures [PW]

References

[Q] Quillen D., Homotopical Algebra, Springer Lecture Notes in Mathematics 43 (1967).

[H] Hovey M., Model Categories, Mathematical Surveys and Monographs Volume 63 (1999).

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[BPhD] Barton R.W., A model 2-category of enriched combinatorial premodel categories, PhD Thesis, Harvard University (2019) https://arxiv.org/abs/2004.12937v1

[DV] Dupont M., Vitale E., Proper factorization systems in 2-categories, Journal of Pure and Applied Algebra Volume 179 Issues 1-2 Pages 65-86 (2003).

[PW] Pronk, D. A., Warren M. A., Bicategorical □fibration structures and stacks, Theory and Applications of Categories 29 (2014).

[DDS] Descotte M.E., Dubuc, E., Szyld M., A localization of bicategories via homotopies, Theory and Applications of Categories, Vol. 35, No. 23, pp 845-874 (2020).

[DDS] Descotte M.E., Dubuc, E., Szyld M., Model bicategories and their homotopy bicategories, https://arxiv.org/abs/1805.07749 (2018) (to be updated soon :)

Before we start: a very basic preliminary

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ functor between categories

ess. full:

$$\begin{array}{ccccc} X' & & FX' & \xrightarrow[a]{\sim} & FX \\ \exists f' \big\downarrow & & \big\downarrow Ff' & & \big\downarrow \forall g \\ Y' & & FY' & \xrightarrow[b]{\sim} & FY \end{array}$$

$b \, Ff' = g a$

(i.e. $\mathcal{C}^2 \longrightarrow (\operatorname{Im} F)^2$ ess surj)

$\downarrow \exists f$ $\qquad\qquad$ $\downarrow g$

The axioms for model bi categories [?] [DDS]

M0. Existence of some bi limits ($\simeq$: there is an inv 2-cell)

M1. Lifting properties

M2. Factorizations

M3. "Stability" of $\mathcal{F}$ (and co$\mathcal{F}$) under

- composition

- (co)base change

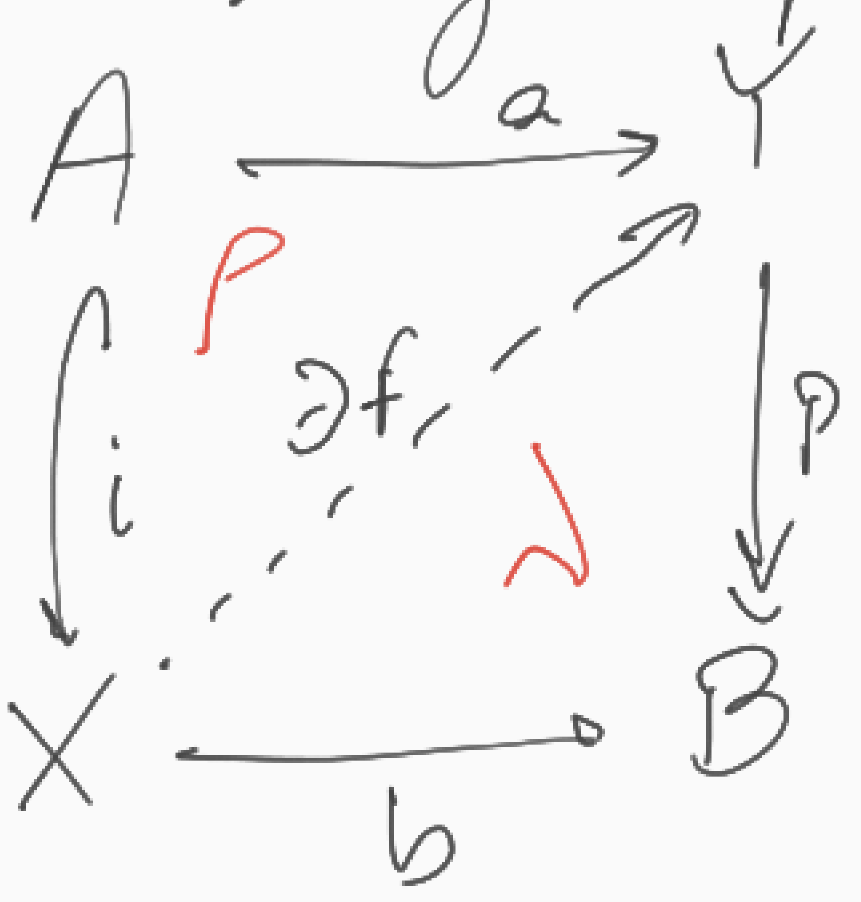
- ~~isos~~ equiv. are $\mathcal{F}$ & co$\mathcal{F}$

M4. Add $\sim$:

M5. W.e. satisfy "3 for 2"

if two are w.e $\Longrightarrow$ so is the third

M1. Lifting Properties



(if either i or p is a w.e.)

γ, ρ, λ

$pa \cong bi \Rightarrow \exists f$ s.t. $pf \cong b$, $fi \cong a$

ρ λ

$$pfi \xRightarrow{\rho i} bi \xRightarrow{\gamma} pa$$

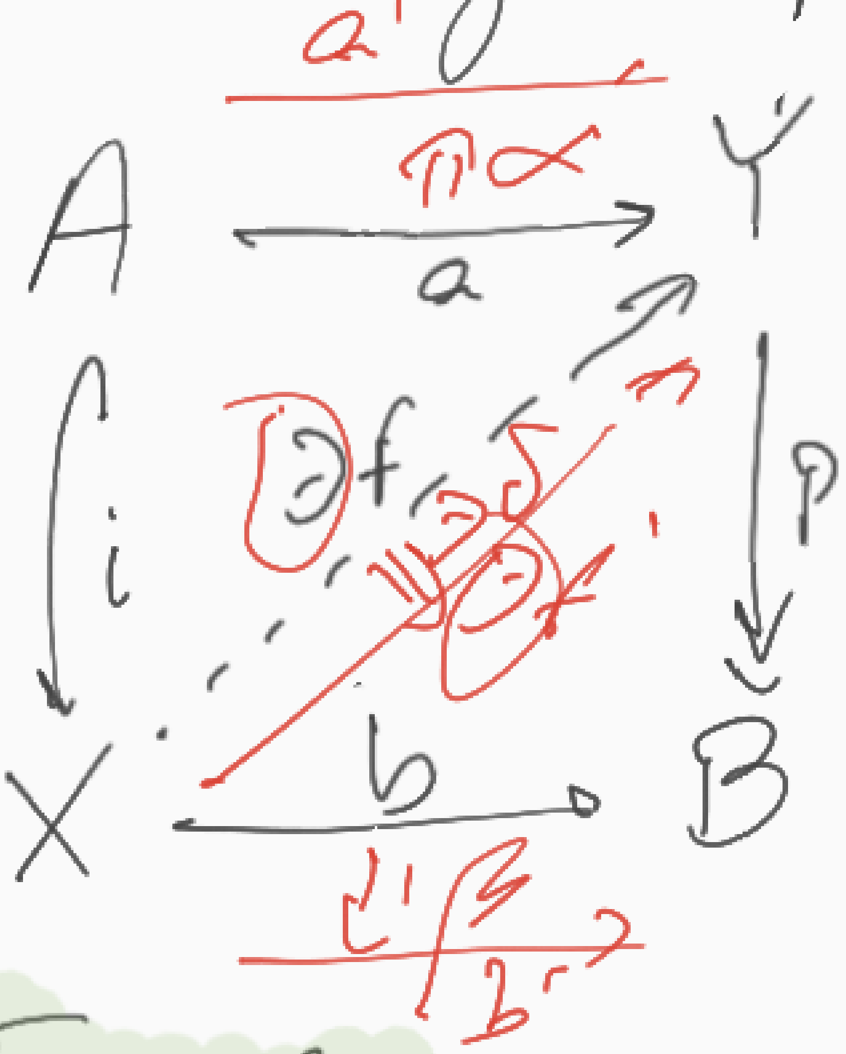
$$= \quad p\lambda$$

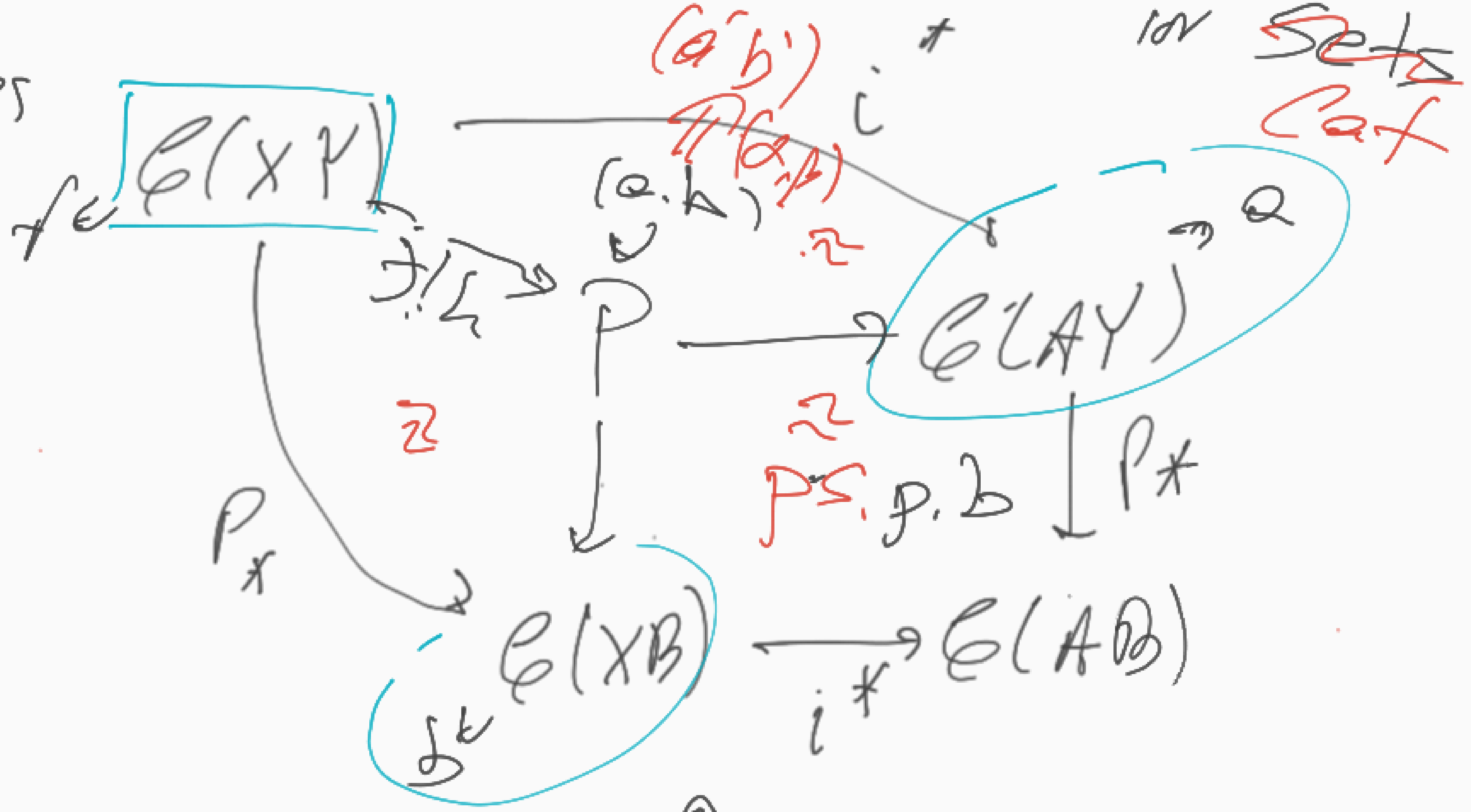
[BPhD]

[BPhD]

"these axioms deal only with the invert. 2-cells of $\mathcal{C}$"

M1. Lifting Properties





Fix i, p

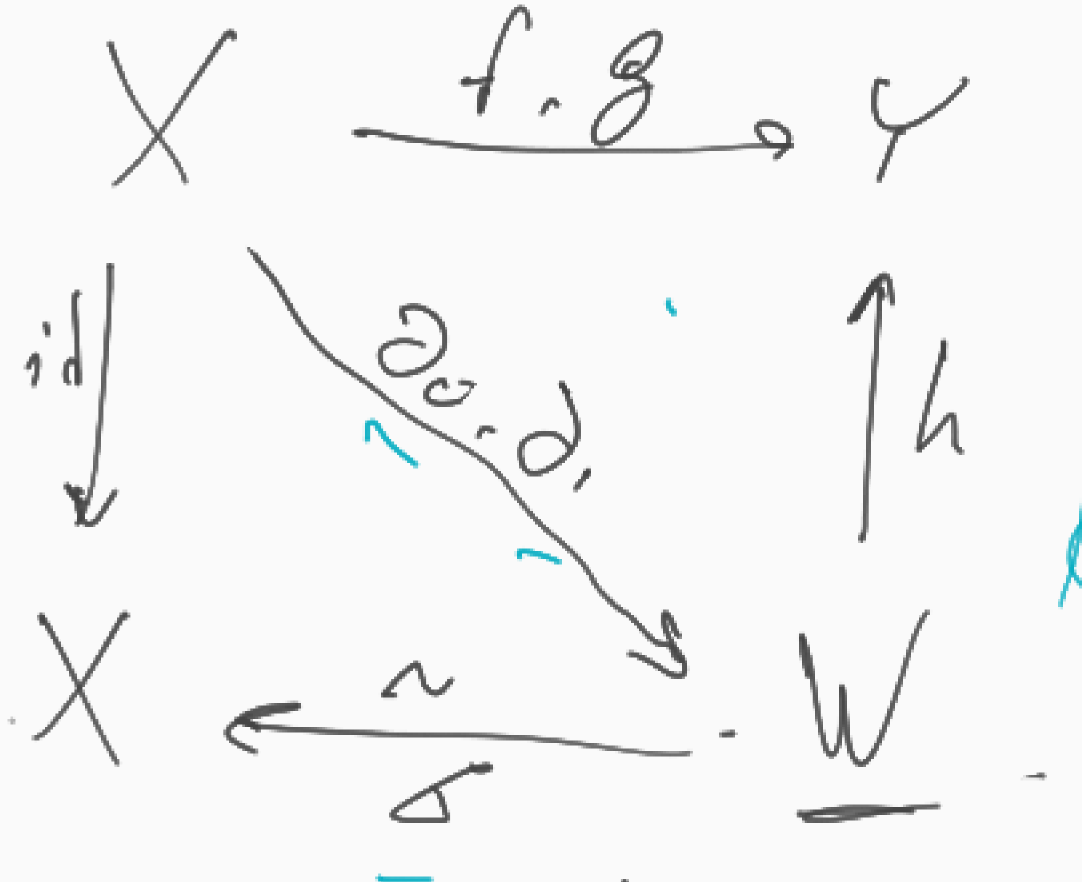
- ∃! f ⟺ the square ⟲ is a p.b. (BPhD)
- ∃ f ⟺ h is surj   ∃(f, β, λ) ⟺ h is ess surj

AXIOM M1: h is ess surj & ess full.

Homotopies

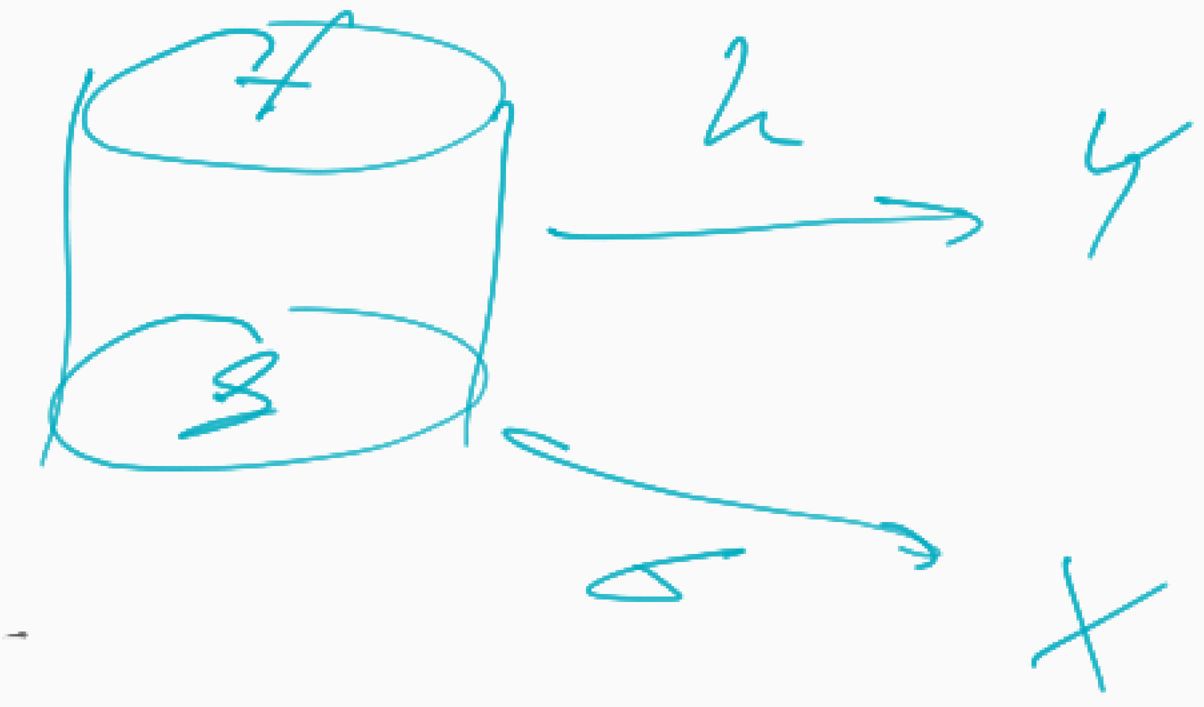
$X \underset{g}{\overset{f}{\rightrightarrows}} Y$

$f \overset{\ell}{\sim} g$:



$f \overset{r}{\sim} g$

"$W = X \times I$"



Basic facts: • $\mathcal{C}_c$: cofibrant objects: $0 \rightarrowtail X$ • $\mathcal{C}_{cf}$

$A \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{p} Y$ • $\mathcal{C}_f$, fibrant objects: $X \twoheadrightarrow 1$

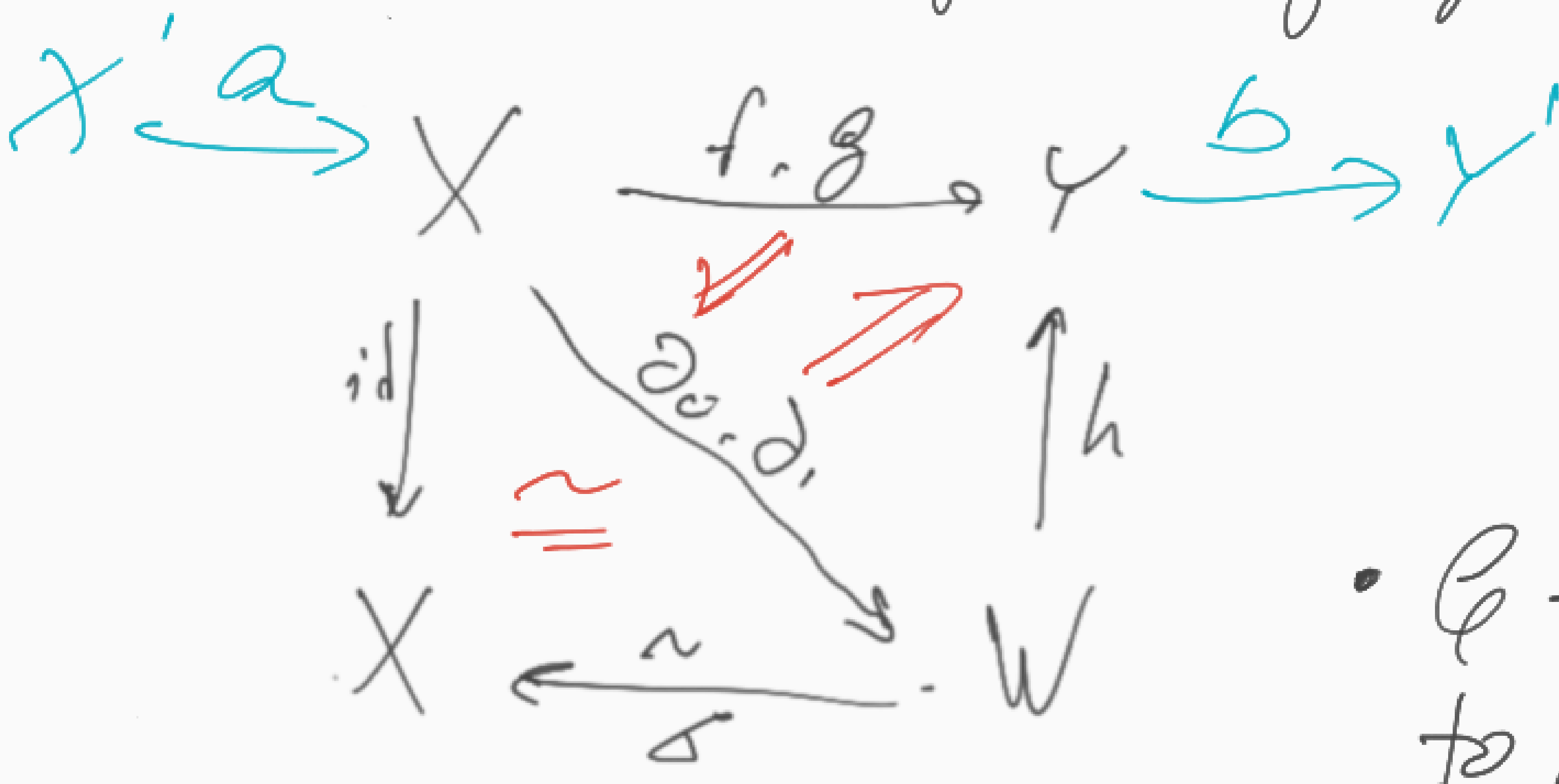
• In $\mathcal{C}_{cf}$, $\overset{\ell}{\sim} = \overset{r}{\sim} = \sim$ & Whitehead holds

$fg \sim id$, $gf \sim id$

• [9, Lemma 7] $A \in \mathcal{C}_c$, $X \overset{\sim}{\underset{p}{\twoheadrightarrow}} Y$ $\Longrightarrow$ $\mathcal{C}(A,X)/\overset{\ell}{\sim} \xrightarrow{p_*} \mathcal{C}(A,Y)/\overset{\ell}{\sim}$

bijection

Towards a homotopy bicategory



$f \overset{\eta}{\Rightarrow} h\partial_0 \qquad h\partial_1 \overset{\varepsilon}{\Rightarrow} g$

$\sigma\partial_0 \underset{\alpha}{\overset{\simeq}{\Rightarrow}} \sigma\partial_1$

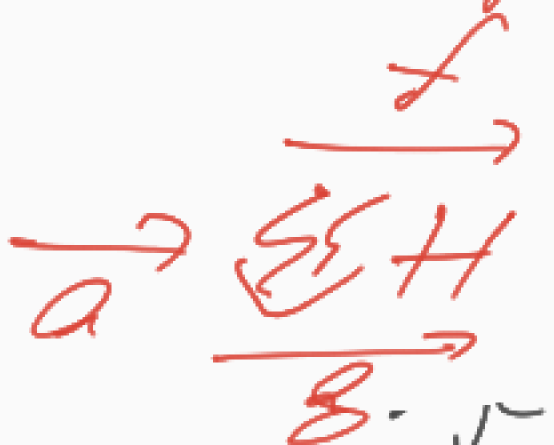
$f \overset{l}{\sim} g \not\Rightarrow f a \overset{l}{\sim} g a$

FOR MODEL CAT

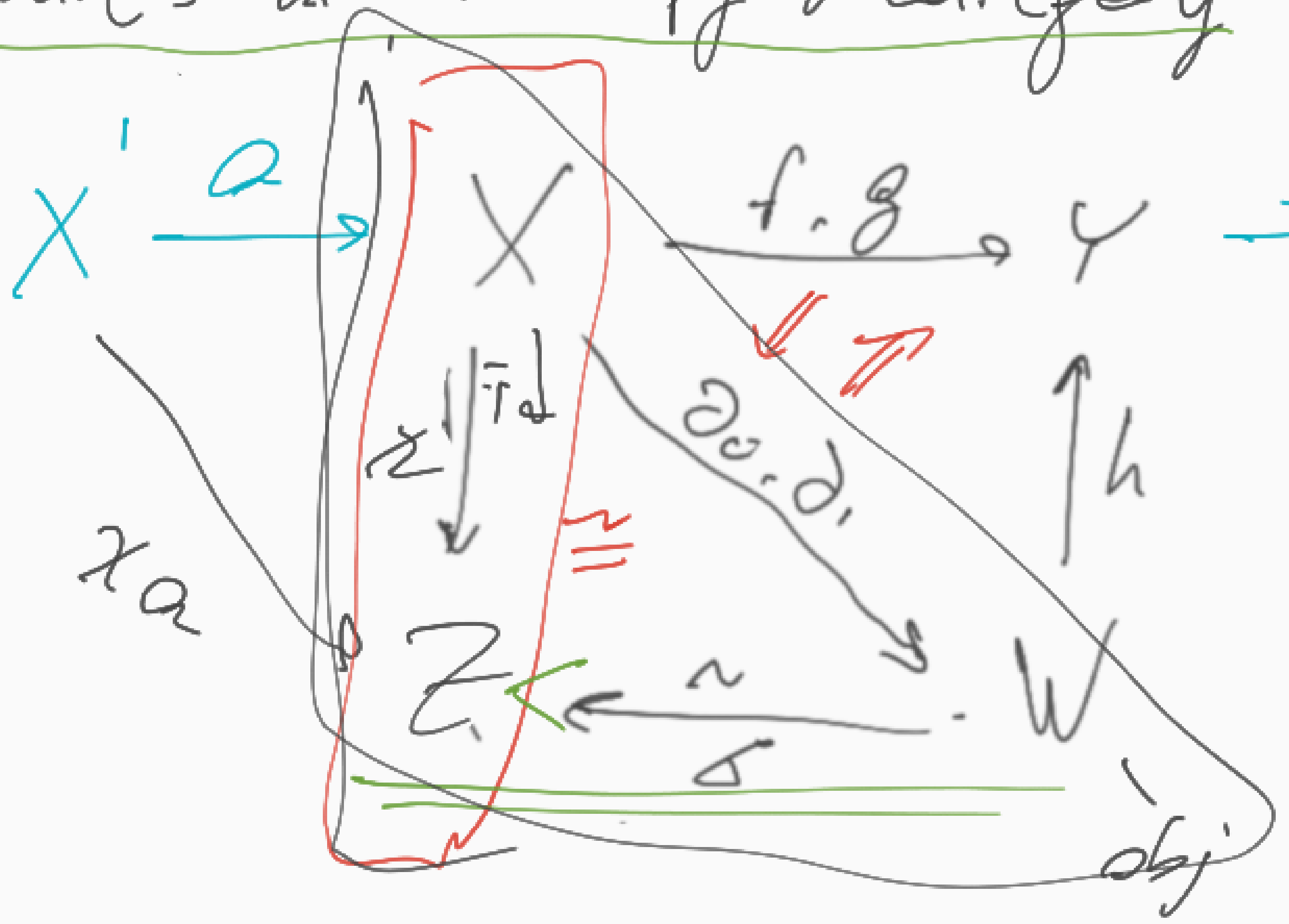
$Ho(\mathcal{C})$ should have

- objects & arrows of $\mathcal{C}$ ($\mathcal{C}_{fc}$)
- 2-cells formed by (classes of) homotopies
- $\mathcal{C} \xrightarrow{i} Ho(\mathcal{C})$: we should be able to build a homotopy $f \rightsquigarrow g$ given $f \overset{\mu}{\Rightarrow} g$

w-homotopies don't seem to form a bicat.



Towards a homotopy bicategory



$H : f \rightsquigarrow g$

$Ha : fa \rightsquigarrow ga$

δ-homotopies

We define a bicategory $Ho^f(\mathcal{C}, W)$ formed with the δ-homot.

$\mathcal{C} \xrightarrow{i} Ho(\mathcal{C}W)$

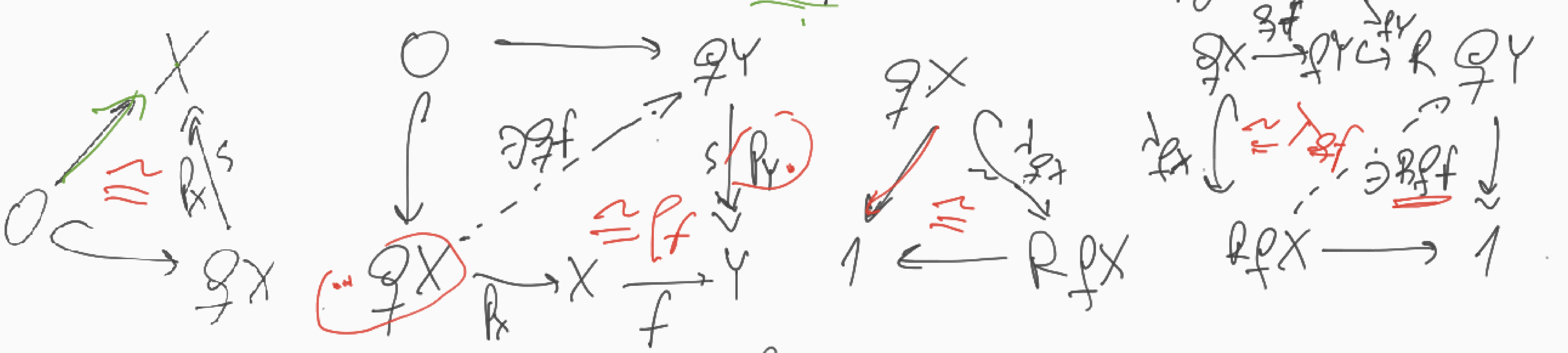
Towards a homotopy bicategory

Basic facts of σ-homotopies (fibrant)

- $\mathcal{H}o^f = \mathcal{H}o^f(\mathcal{C}W)$ bicategory, $\mathcal{C} \xrightarrow{i} \mathcal{H}o^f$, $\mathcal{C} \xrightarrow{\gamma_F} \mathcal{D}$ "1/2 u.p." $FW \subseteq \{eqs\}$
- In $\mathcal{C}_{fc}$ "$\sigma = W$" & whitehead holds $\rightsquigarrow \mathcal{H}o(\mathcal{C})$ obj $/\mathcal{C}_{fc}$
- $A \in \mathcal{C}_c$, $X \overset{p}{\underset{\sim}{\twoheadrightarrow}} Y$) $\mathcal{H}o^f(AX) \xrightarrow{p_*} \mathcal{H}o^f(AY)$ equiv. of categ.. "

§ 7

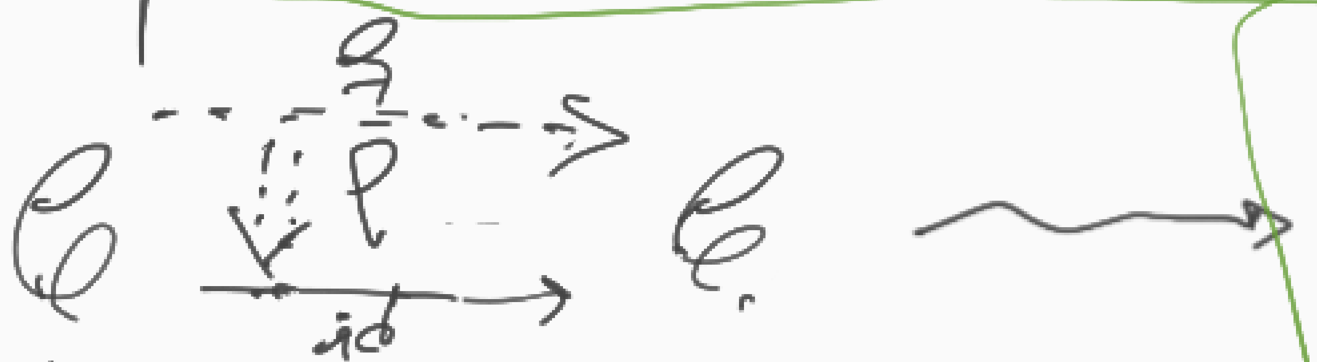
Replacement $\mathcal{C} \overset{Q}{\dashrightarrow} \mathcal{C}_c \overset{R}{\dashrightarrow} \mathcal{C}_{cf}$ → Recall: good notion of homotopy & Whitehead

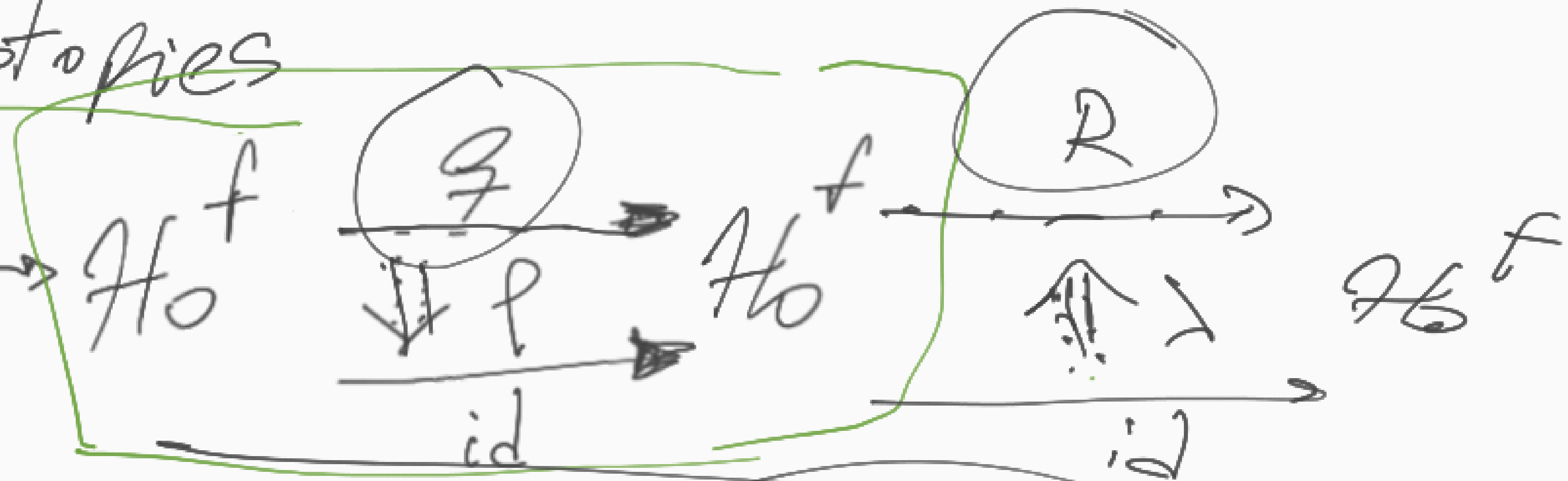


- this is NOT ~~pseudo~~ functorial
- Qf is "only" determined up to left homotopy [By 7]
- this defines $\mathcal{C} \longrightarrow Ho(\mathcal{C})$ the localization of $\mathcal{C}$ at the w.e.

  $f \longmapsto [RQf]$

The replacement on homotopies

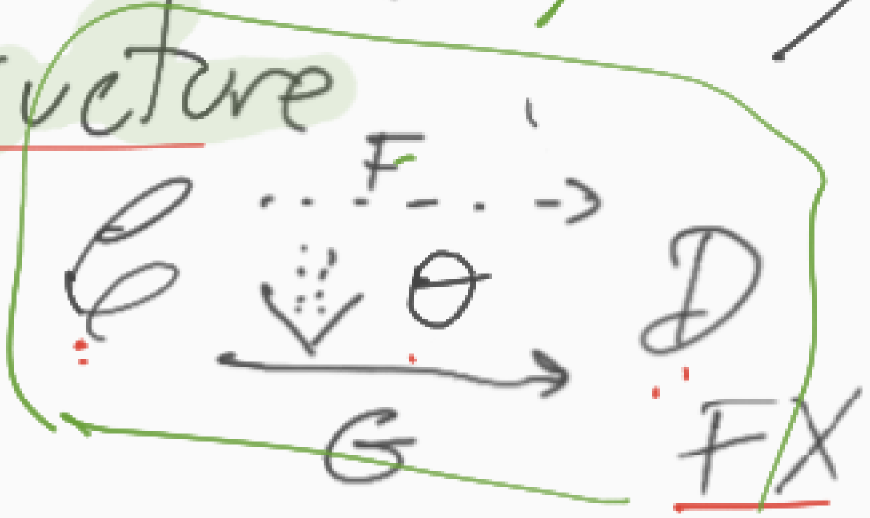


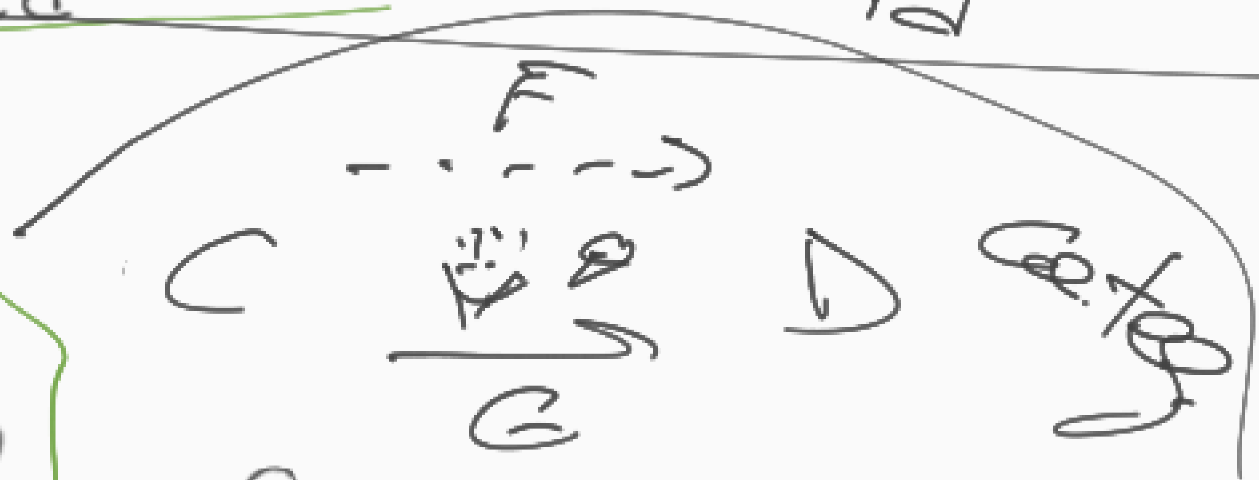


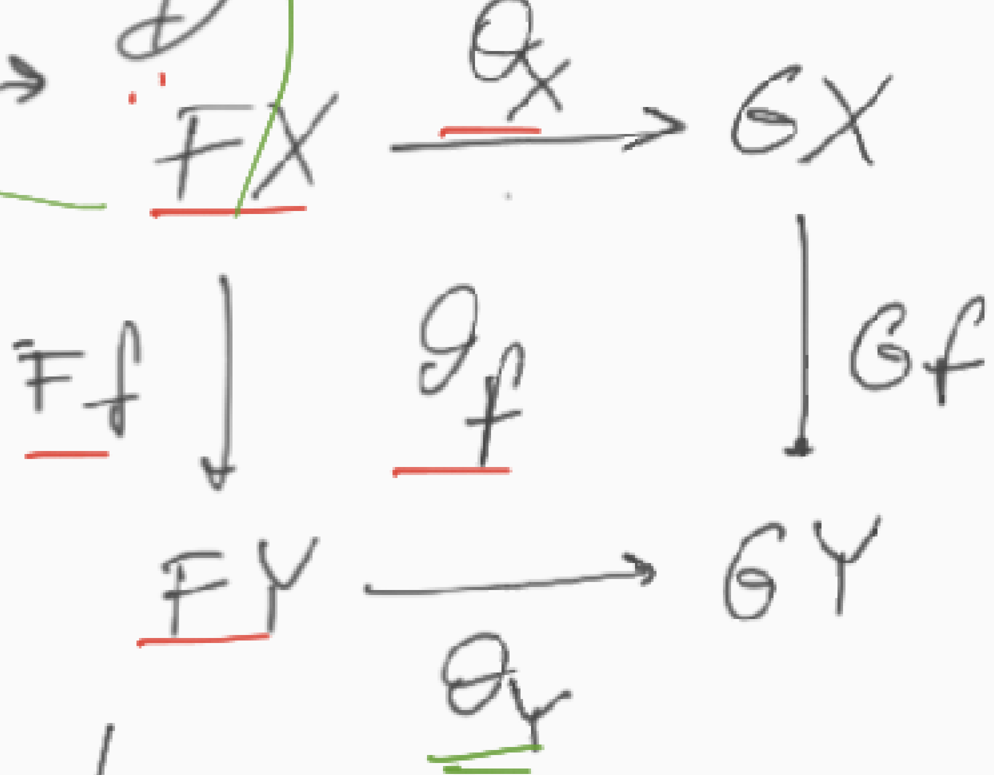
"assignations" gX, gf

ρX, ρf

by "transfer of structure" (for bi-categories)







$$\mathcal{D}(FX,FY) \xrightarrow{(\theta_Y)_*} \mathcal{D}(FX\,GY)$$

equiv. of categ

$\Longrightarrow \exists!$ pseudofunctor structure for F s.t. θ ps. nat

The localization result

$$\begin{array}{ccc} \mathcal{C} & \overset{RQ}{\dashrightarrow} & \mathcal{C}_{fc} \\ {\scriptstyle i}\downarrow & \overset{RQ}{=} & \downarrow{\scriptstyle i} \\ Ho^f(\mathcal{C}W) & \xrightarrow{\ r\ } & Ho(\mathcal{C}) \end{array}$$

$Ho(\mathcal{C})$: obj & arrows of $\mathcal{C}_{fc}$

2-cells = classes of homotopies (remember $\sigma = w$ here)

Theorem: $\mathcal{C} \xrightarrow{ri} Ho(\mathcal{C})$ is the (bicat.) localization of a model bicategory $\mathcal{C}$ at the weak equivalences. It is locally small (resp. a 2-category) when $\mathcal{C}^{(\text{fixed cylinder})}$ is so.

Thank You!