The structure (and story) of ω-complete effect monoids

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This talk

1. What are effect monoids? and why would you want to consider the ω-complete ones.
2. Origin of effect monoids: the scalars of an effectus
3. Representation theory for ω-complete effect monoids
4. Duality for directed complete effect monoids

Effect monoids

Examples:

B, Boolean algebra; $[0,1]_{\mathscr{A}} = \{a \in \mathscr{A} : 0 \leqslant a \leqslant 1\}$,
$\mathscr{A}$ commutative unital C^*-algebra

That is: $C(X, \{0,1\})$; $C(X, [0,1])$, X compact Hausdorff

Definition: An **effect monoid** is a set M with a

1. **partial addition** $\ovee$,
   $a \ovee b := a \vee b$, when $a \wedge b = 0$ | $a \ovee b := a + b$, when $a + b \leqslant 1$
2. a **complement** operation $(\,\cdot\,)^\perp$,
   $a^\perp := \neg a$ | $a^\perp := 1 - a$
3. a **zero** (and **one**) element 0 (and $1 := 0^\perp$),
4. and a **multiplication** $\cdot$,
   $a \cdot b := a \wedge b$ | regular multiplication

obeying certain axioms (next slide).

Effect monoid axioms

1. $a \ovee b = b \ovee a$
2. $(a \ovee b) \ovee c = a \ovee (b \ovee c)$
3. $a \ovee 0 = a$
4. $a \ovee a^{\perp} = 1$
5. $a \ovee b_1 = a \ovee b_2$ implies $b_1 = b_2$
6. $a \ovee b = 0$ implies $a = b = 0$
7. $1 \cdot a = a = a \cdot 1$
8. $(ab)c = a(bc)$
9. $a(b \ovee c) \rightharpoonup ab \ovee ac$ & $(b \ovee c)a \rightharpoonup ba \ovee ca$

$A \rightharpoonup B$ means "when A is defined, so is B, and they're equal."

$\frac{1}{2} \cdot \frac{2}{3} \ovee \frac{1}{2} \cdot \frac{2}{3}$ makes sense in $[0, 1]$, but $\frac{1}{2}(\frac{2}{3} \ovee \frac{2}{3})$ doesn't.

Dropping axioms 7–9 and $\cdot$ we get an **effect algebra**.

Examples of effect monoids

1. **Boolean algebras**
2. $[0,1]_{\mathscr{A}}$, where $\mathscr{A}$ is a **commutative unital C^*-algebra**.
   (Commutative, because $ab \geqslant 0$ for $a, b \geqslant 0$ implies $ab = (ab)^* = b^* a^* = ba$.
   For non-commutative C^*-algebras, the 'sequential product' $a \& b := \sqrt{a} b \sqrt{a}$ can be restricted to $[0,1]_{\mathscr{A}}$, leading to Gudder's 'sequential effect algebras'.)
3. $[0,1]_R$, where R is a **partially ordered** (not necessarily commutative) **unital ring** R.
   (A Boolean algebra B seen as ring with 'xor' $a \oplus b := (a \vee b) \wedge \neg(a \wedge b)$ as addition is not partially orderable, since $a \oplus a = 0$.)

A non-commutative effect monoid[1]

$[0,1]_R$, where R is the totally ordered unital ring on the vector space $\mathbb{R}^5$, ordered lexicographically, i.e.

$$v < w \iff \exists N < 5\,[\, v_N < w_N \;\wedge\; \forall\, n < N\,[\, v_n = w_n \,]\,],$$

with multiplication given on basis vectors $e_1 = (1,0,0,0,0), \ldots, e_5 = (0,0,0,0,1)$ by:

$\cdot$	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5
e_2	e_2	e_4	e_5	0	0
e_3	e_3	0	0	0	0
e_4	e_4	0	0	0	0
e_5	e_5	0	0	0	0

(Totally ordered non-commutative rings can not be Archimedean.)

[1] From Bas Westerbaan's master's thesis.

Effect monoids are terrible structures

1. Only trivial things can be proven about them,
2. and obvious propositions seem to be false,
   (e.g. that $aa^\perp \ovee aa^\perp \ovee aa^\perp$ exists.)
3. though counterexamples are difficult to obtain,
4. but give no deep insight when found.

The situation is completely different for ω-**complete** effect monoids!

ω-completeness

In an effect monoid (or effect algebra) M we define:

$$a \leqslant b \iff \exists d \in M.\ b = a \ovee d.$$

(By the way, such a d is unique when it exists, and written $b \ominus a$.)

An effect monoid is ω-**complete** when every ascending sequence $a_1 \leqslant a_2 \leqslant \cdots$ has a supremum $\bigvee_n a_n$.
(We do not require $\ovee$ and $\cdot$ to preserve these suprema.)

Examples of ω-complete effect monoids (ω-EMs)

1. ω-complete Boolean algebra, such as a σ-algebra.
2. $[0,1]$-valued measurable functions on a σ-algebra.
3. $[0,1]_{\mathscr{A}}$, where $\mathscr{A}$ is a 'ω-complete' commutative unital C^*-algebra (such as a commutative von Neumann algebra.)
4. $C(X,[0,1])$ where X is a compact Hausdorff is ω-complete iff X is **basically disconnected**, that is, $\overline{X \backslash f^{-1}(0)}$ is open for every $f \in C(X,[0,1])$.
5. The clopens $C(X,\{0,1\})$ of a basically disconnected compact Hausdorff space X.

ω-complete effect monoids are great structures!

Given an ω-EM M.

1. One easily sees that M has **no infinitesimals**:
   if na exists for all n, then $a \ovee \bigovee_{n=0}^{\infty} a = \bigovee_{n=0}^{\infty} a$, so $a = 0$.
2. Harder: M can be **represented by continuous functions**, that is, is isomorphic to a subalgebra of $C(X, [0,1])$ for some basically disconnected compact Hausdorff space X.
3. In particular, M is **commutative**.
4. **Lattice:** binary infima $a \wedge b$ and suprema $a \vee b$ exist.
5. Above each $a \in M$ there is a least **idempotent** $\lceil a \rceil$.
6. **division:** For all $a \leqslant b$ we can define $a/b \in M$ with $a = (a/b)b$.
7. Multiplication **preserves all existing suprema** (not just countable directed ones.)

Directed complete effect monoid

A **directed complete effect monoid** (**dcEM**) is isomorphic to

$$C(X, [0,1]) \oplus C(Y, \{0,1\})$$

for *extremally disconnected* (the closures of opens are open) compact Hausdorff spaces X and Y.

So a directed complete effect monoids splits into **Boolean** and **convex** parts.

(This can be turned into a categorical duality, as we'll see later.)

Counterexample

ω-**EMs don't split** in Boolean and convex parts:

Consider, given an uncountably infinite set X, the ω-EM

$$M := \left\{ f\colon X \to [0,1] \colon \left[\begin{array}{l} \text{either } f^{-1}(0) \text{ is cocountable} \\ \text{or } f^{-1}(1) \text{ is cocountable} \end{array} \right. \right\}$$

(So each $f \in M$ is either mostly equal to 0, or mostly equal to 1.)

Note that M has no Boolean idempotents, no half, but does have a maximal set of orthogonal halvable idempotents.

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Origin of effect monoids: effectuses

An **effectus** is a category E with finite coproducts, final object 1, such that

$$
\begin{array}{ccc}
A + B & \xrightarrow{1+!} & A + 1 \\
{\scriptstyle !+1}\downarrow & & \downarrow{\scriptstyle !+1} \\
1 + B & \xrightarrow[1+!]{} & 1 + 1
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{!} & 1 \\
{\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\
A + B & \xrightarrow[!+!]{} & 1 + 1
\end{array}
$$

are pullbacks, and

$$[\kappa_1,\kappa_2,\kappa_2],\ [\kappa_2,\kappa_1,\kappa_2] : 1 + 1 + 1 \longrightarrow 1 + 1$$

are jointly monic.

States and predicates

Origin of effect monoids: effectuses

An effectus is intended to reason about **states** $s\colon 1 \to X$, **predicates** $p\colon X \to 1+1$, (and **partial maps** $f\colon X \to Y+1$.)

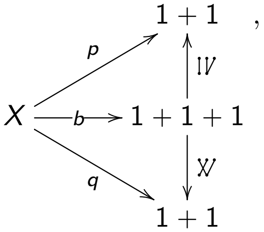
The composition $p \circ s$ is a morphism $1 \to 1+1$ (called a scalar) that represents the probability that predicate p holds in state s.

It's the morphisms $1 \to 1+1$ that form an effect monoid.

Addition of predicates

Origin of effect monoids: effectuses

Predicates $p, q \colon X \to 1 + 1$ (in particular, scalars $1 \to 1 + 1$) in an effectus are **summable** when there is a $b \colon X \to 1 + 1 + 1$ with



in which case we define $p \ovee q := \nabla \circ b$.

(Note that b is unique by joint monicity of ⩗ and ⩘.)

Predicates form an effect algebra

Origin of effect monoids: effectuses

The predicates $X \to 1 + 1$ form an effect algebra with:

$$1 \;:=\; \big(\, X \xrightarrow{\;!\;} 1 \xrightarrow{\;\kappa_1\;} 1+1 \,\big)$$

$$0 \;:=\; \big(\, X \xrightarrow{\;!\;} 1 \xrightarrow{\;\kappa_2\;} 1+1 \,\big)$$

$$p^{\perp} \;:=\; \big(\, X \xrightarrow{\;p\;} 1+1 \xrightarrow{\;\text{X}\;} 1+1 \,\big)$$

For example, $p \ovee p^{\perp} = 1$, because

$$b := \big(\, X \xrightarrow{\;p\;} 1+1 \xrightarrow{\;\text{II.}\;} 1+1+1 \,\big)$$

satisfies $\quad \text{IIV} \circ b = p, \qquad \text{XV} \circ b = p^{\perp}, \qquad \text{VV} \circ b = 1.$

Multiplication of scalars

Origin of effect monoids: effectuses

The scalars $1 \to 1 + 1$ form an effect monoid with multiplication:

$$s \cdot t \;:=\; \big(\, 1 \xrightarrow{\;s\;} 1 + 1 \xrightarrow{\;t+1\;} 1 + 1 + 1 \xrightarrow{\;\mid\!\vee\;} 1 + 1 \,\big)$$

(Which is, if you like, the Kleisli composition of s and t with respect to the monad $(\,\cdot\,) + 1$ that has unit $X \xrightarrow{\kappa_1} X + 1$ and multiplication $X + 1 + 1 \xrightarrow{\mid\!\vee} X + 1$.)

Note that there is no reason to expect that this multiplication is commutative. In fact, any effect monoid M occurs as the scalars of some effectus (for example, the effectus of 'effect modules' over M.)

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Idempotents

An element p of an effect monoid M is an **idempotent** when $p^2 = p$, that is, $pp^\perp = 0$.

Given $a \in M$, we have:

1. $a \leqslant p \iff ap^\perp = 0 \iff ap = a$.
2. $ap = pap = pa$

(So all idempotents are 'central'.)

Corollary: pM is an effect monoid called (with unit p) called a **corner**, and $M \cong pM \oplus p^\perp M$ via $a \mapsto (pa, p^\perp a)$.

Boolean and halvable idempotents

An idempotent p of an effect monoid M is

1. **Boolean** when all $a \leqslant p$ are idempotents;
2. **halvable** when there is $a \in M$ with $a \ovee a = p$.

We say that M is Boolean/halvable when 1_M is Boolean/halvable.

It turns out that:

1. An effect monoid is Boolean iff M is a Boolean algebra (easy, because the idempotents form a Boolean algebra).
2. An ω-EM M is halvable iff $M \cong C(X, [0,1])$ for some basically disconnected compact Hausdorff space X (hard—we'll get back to this.)

How to get (Boolean and halvable) idempotents?

Given an element a of an effect monoid M we have

$$1 = a \ovee a^\perp = a \ovee (a \ovee a^\perp)a^\perp = a \ovee aa^\perp \ovee (a^\perp)^2 = \cdots .$$

Going on like that, we get:

$$1 \;=\; \textstyle\bigovee_{n<N} a(a^\perp)^n \;\ovee\; (a^\perp)^N.$$

So when M is ω-complete, we can define

$$\lceil \boldsymbol{a} \rceil \;:=\; \textstyle\bigovee_{n<N} a(a^\perp)^n \qquad \text{and} \qquad \lfloor \boldsymbol{a} \rfloor \;:=\; \textstyle\bigwedge_n a^n.$$

Then $\lfloor a \rfloor$ is the greatest idempotent below a (intuitively, because c^n converges to 0 when $c \in [0,1)$) and $\lceil a \rceil$ is the least idempotent above a.

Non-trivial property of ceiling

Proposition: $ab = 0 \implies a\lceil b\rceil = 0$.

Proof: One hopes that $a\lceil b\rceil \equiv a \bigovee_n b(b^\perp)^n \stackrel{?}{=} \bigovee_n (ab)(b^\perp)^n = 0$, but does $a(\,\cdot\,)$ preserve suprema? (We only automatically have $\geqslant$.)

Writing $s_N := \bigovee_{n=0}^N b(b^\perp)^n$, we have $as_N = 0$, so $a^\perp s_N = s_N$, so

$$\lceil b\rceil \;=\; \bigvee_N s_N \;=\; \bigvee_N a^\perp s_N \;\leqslant\; a^\perp \bigvee_N s_N \;\equiv\; a^\perp\lceil b\rceil \;\leqslant\; \lceil b\rceil$$

Thus $a^\perp\lceil b\rceil = \lceil b\rceil$, so $a\lceil b\rceil = 0$.

How to get halvable idempotents

When $b \equiv a \ovee a$, then $\lceil b \rceil = (\bigovee_n a(b^\perp)^n) \ovee (\bigovee_n a(b^\perp)^n)$ is an halvable idempotent.

How to get halvable elements? Given a, we have $aa^\perp = a^\perp a$, because adding a^2 to either side gives a.
Now, $1 = (a \ovee a^\perp)^2 = a^2 \ovee 2aa^\perp \ovee (a^\perp)^2$, so $2aa^\perp$ exists.

Whence $\lceil 2aa^\perp \rceil$ is a halvable idempotent (but might be zero.)
If $\lceil 2aa^\perp \rceil = 0$, then $aa^\perp \leqslant \lceil 2aa^\perp \rceil = 0$, so a is an idempotent.

Moral: When the halveable idempotents/elements of an ω-EM are exhausted, only (Boolean) idempotents are left.

Crux of the representation theorem

Using Boolean and halvable idempotents: Let E be a maximal set of orthogonal idempotents of an ω-EM M such that each $p \in E$ is either Boolean or halvable, then it turns out (we'll get back to this) that the map

$$\varrho\colon a \mapsto (pa)_{p\in E}\colon \;\; M \longrightarrow \bigoplus_{p\in E} pM$$

is an isomorphism onto its image.
(We cannot always expect surjectivity.)

Each pM, being Boolean or halvable, will turn out to be isomorphic to either a $C(X,\{0,1\})$ or a $C(X,[0,1])$ for some basically disconnected compact Hausdorff space X.

Counterexample

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From halvable ω-EM to $C(X, [0,1])$, I

Given a halvable ω-EM M and $h \in M$ with $h \ovee h = 1$, we can define a **scalar multiplication** $[0,1] \times M \to M$ first on the dyadics by $\frac{m}{2^n} \cdot a = mh^n a$, and then extend it to all $[0,1]$ by ω-completeness, such that:

1. $\lambda(\mu a) = (\lambda\mu)a$
2. $(\lambda \ovee \mu)a \rightleftharpoons \lambda a \ovee \mu a$
3. $\lambda(a \ovee b) \rightleftharpoons \lambda a \ovee \lambda b$
4. $1 \cdot a = a$

That is, M is a '**convex effect algebra**'.

From halvable ω-EM to $C(X, [0,1])$, II

Such a convex effect algebra is, by a theorem of Gudder and Pulmannová's, isomorphic to $[0,1]_V$ for some **order unit space** V (i.e. partially ordered real vector space with a positive element 1 such that for each $a \in V$ there is n with $-n \leqslant a \leqslant n$.)

However: to see that $V \cong C(X)$ we need additional structure:

- either extend multiplication to V (and use Kadison's representation theory for ordered algebras);
- or show M (and thus V) is a lattice (and use Yosida's representation theorem for vector lattices.)

We'll go for the lattice structure.

Lattice structure, I

$A+B = A \vee B + A \wedge B$

$A \wedge B \leq C \leq A \vee B$

Idea: approximate $a \wedge b$ using multiplication.

First approximation: ab.

Note in $[0, 1]$ (and so in $C(X, [0, 1])$ too)

$$a \wedge b \ominus ab \;=\; (a \ominus ab) \wedge (b \ominus ab).$$

Second approximation: $ab \circledvee (a \ominus ab)(b \ominus ab))$.

Going on like this. . .

Lattice structure, II

Given elements a and b of an ω-complete effect monoid M, define

$$a \wedge b \;:=\; \bigovee_{n=0}^{\infty} a_n b_n \quad \text{where} \quad \left[\begin{array}{l} a_N := a \ominus \bigovee_{n<N} a_n b_n \\ b_N := b \ominus \bigovee_{n<N} a_n b_n \end{array}\right.$$

Then $a \wedge b$ is the infimum of a and b.

(Note that $a_N b_N$ is summable with $\bigovee_{n<N} a_n b_n$, because $a_N \geqslant a_N b_N$ is, by definition; moreover, $a_N \ovee \bigovee_{n<N} a_n b_n = a$, implies $\bigovee_{n\leqslant N} a_n b_n \leqslant a$.)

We also get:

$$\bigwedge_n a_n \;:=\; a \ominus a \wedge b \qquad \text{and} \qquad \bigwedge_n b_n \;:=\; b \ominus a \wedge b.$$

Lattice structure, III

So why is $a \wedge b$ the infimum? Clearly, $a \wedge b \leqslant a, b$.
So let $\ell \leqslant a, b$ be given; we must show that $a \wedge b \geqslant \ell$.

Note that $(\bigwedge_n a_n)(\bigwedge_n b_n) \leqslant \bigwedge_n a_n b_n = 0$, because $N \bigwedge a_n b_n$ exists for all N, because $\bigovee_N a_n b_n$ exists.
Writing $\boldsymbol{p} := \lceil \bigwedge_n \boldsymbol{b_n} \rceil$, we have:

$$(\bigwedge_n a_n)p = 0 \qquad \text{and} \qquad (\bigwedge_n b_n)p^\perp = 0.$$

Thus, as $a \wedge b \ovee \bigwedge_n b_n = b$,

$$bp^\perp = (a \wedge b)p^\perp \qquad \text{and similarly} \qquad ap = (a \wedge b)p.$$

Now, $\ell = \ell p^\perp \ovee \ell p \leqslant bp^\perp \ovee ap = (a \wedge b)p^\perp \ovee (a \wedge b)p = a \wedge b$.

Whence: $a \wedge b$ is the greatest lower bound of a and b.

From halvable ω-EM to $C(X,[0,1])$, III

Getting back to our halvable ω-EM M that is isomorphic to $[0,1]_V$ for some order unit space V:

- Since M is a lattice, so is V;
- Since M is ω-complete, so is V, for bounded sequences.

Whence V is a 'σ-**Dedekind complete Riesz space**', and thus, by Yosida's representation theorem, isomorphic to $C(X)$ for some basically disconnected compact Hausdorff space.

(As a result, $M \cong C(X,[0,1])$.)

From halvable ω-EM to $C(X, [0, 1])$, IV

In more detail: We have a *Riesz space isomorphism* (linear, unital, and $\wedge$-preserving)

$$a \mapsto (\varphi \mapsto \varphi(a)) \colon \; V \longrightarrow C(\mathrm{Spec}(V)),$$

where $\mathrm{Spec}(V)$ is the basically disconnected compact Hausdorff subspace of $\mathbb{R}^V$ consisting of the Riesz homomorphisms $\varphi \colon V \to \mathbb{R}$.

This map restricts to an isomorphism $M \to C(\mathrm{Spec}(V), [0, 1])$, of lattice effect algebras.

But does this isomorphism preserve multiplication?

From halvable ω-EM to $C(X,[0,1])$, V

For $a \mapsto (\varphi \mapsto \varphi(a))\colon M \longrightarrow C(\mathrm{Spec}(V),[0,1])$ to **preserve multiplication**, it suffices to show that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in M$ and $\varphi \in \mathrm{Spec}(V)$.

Proof: define $d(a,b) := a \vee b \ominus a \wedge b$. Then:

1. $d(ab, ac) \leqslant a\, d(b,c) \leqslant d(b,c)$,
   since $ab \vee ac \leqslant a(b \vee c)$ and $ab \wedge ac \geqslant a(b \wedge c)$.
2. $d(a,b) = 0 \implies a = b$,
   since a and b are squeezed between $a \wedge b$ and $a \vee b$.
3. $\varphi(d(a,b)) = d(\varphi(a), \varphi(b))$, since φ preserves $\wedge$, $\vee$ and $\ominus$.

Now: $d(\,\varphi(ab), \varphi(a)\varphi(b)\,) \;=\; \varphi(d(\,ab, a\varphi(b)\,)) \;\leqslant\; \varphi(\,d(b, \varphi(b)1)\,) \;=\; d(\varphi(b), \varphi(b)) = 0$, thus $\varphi(ab) = \varphi(a)\varphi(b)$.

So indeed, halvable ω-EMs are $C(X,[0,1])$-es!

Crux of the representation theorem, II

The key to proving that given an maximal orthogonal set of idempotents E of an ω-complete effect monoid M the map

$$\varrho\colon a \mapsto (pa)_{p\in E}\colon M \longrightarrow \bigoplus_{p\in E} pM$$

is an embedding is showing that ϱ reflects the order:

$$a \leqslant b \quad \Longleftarrow \quad \forall p \in E\,[\, pa \leqslant pb \,].$$

For this, it suffices to show that $a = \bigvee_{p\in E} pa$.

We can get this using $\bigvee E = 1$ (because E is maximal) if multiplication preserves arbitrary suprema:

$$a \;=\; 1\cdot a \;=\; (\bigvee E)a \;=\; \bigvee_{p\in E} pa.$$

Preservation of suprema

How does one prove an operation preserves suprema?

Example: Given an element a of an ordered vector space V,

$$a + (\,\cdot\,)\colon V \to V \qquad \text{is not only order preserving,}$$

but also has an order preserving inverse $(\,\cdot\,) - a\colon V \to V$, so $a + (\,\cdot\,)$ is an **order isomorphism**, and therefore preserves suprema (and infima.)

Division

Similarly, multiplication in an ω-EM M preserves suprema, essentially because of the existence of a (partially defined) **division**:

Proposition: Given $a \leqslant b$ in M,

$$a/b \ := \ \bigovee_{n=0}^{\infty} a(b^{\perp})^n$$

satisfies $(a/b)b = a$, and provides an order preserving inverse to the map $a \mapsto ab \colon M[b] \to Mb$.

P.S. note that $a/a = [a]$. (So $0/0 = 0$ in this context.)

Representation theorem, for ω-EMs

All in all, given an ω-EM M we get an embedding

$$M \longrightarrow \bigoplus_{p \in E} pM \cong C(X, [0,1]) \oplus C(Y, \{0,1\}),$$

(by aggregating the Boolean and halvable factors) where X and Y are basically disconnected compact Hausdorff spaces.

In particular, M is commutative.

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Representation theory, for dcEMs

A directed complete effect monoid (dcEM) M properly splits in **'discrete'** and **'continuous'** parts:

$$M \;\cong\; eM \oplus e^{\perp}M \;\cong\; C(X,[0,1]) \oplus C(Y,\{0,1\}),$$

where X and Y are extremally disconnected compact Hausdorff spaces.
(Indeed, in M there will by directed completeness be a greatest halvable idempotent e, and $e^{\perp}$ will necessarily be Boolean.)

Spectrum of effect monoids

To turn this into a duality, define the **spectrum** of a dcEM M by:

$$\mathrm{Spec}(M) \;:=\; \{\, \varphi\colon M \to [0,1] \text{ preserving } \ovee,\ 1,\ \cdot \,\}.$$

The **discrete part** of the spectrum is defined to be:

$$D_M \;:=\; \{\, \varphi \in \mathrm{Spec}(M)\colon \varphi(M) \subseteq \{0,1\} \,\}.$$

Then D_M is an clopen of the extremally disconnected compact Hausdorff space $\mathrm{Spec}(M)$.

Conversely, given a clopen C of a extremally disconnected compact Hausdorff space X we get a directed complete effect monoid

$$C(X, D) \;:=\; \{\, f\colon X \to [0,1] \text{ continuous with } f(D) \subseteq \{0,1\} \,\}.$$

Duality

The operations give a **duality** between:

- the category of directed complete effect monoids with effect monoid homomorphisms (that preserve $\ovee$, 1, $\cdot$, but not necessarily arbitrary suprema); and
- the category with clopen subsets of extremally disconnected compact Hausdorff spaces,
  with as morphisms from $D_1 \subseteq X_1$ to $D_2 \subseteq X_2$ the continuous maps $f\colon X_1 \to X_2$ with $f(D_1) \subseteq D_2$.

That's it!

Questions?

Some applications:

1. Study of convexity in sequential effect algebras
   (JvdW, BW, AW, arXiv:2004.12749v2)
2. 'ω-complete' effectuses
   (Kenta Cho, JvdW, BW, arXiv:2003.10245)
3. Reconstructing quantum theory without positing real probabilities (Part A of JvdW's thesis, arXiv:2101.03608)