CALCULATING THE CURVATURE TENSOR FOR THE 3D GODEL-LIKE SPACETIMES

Abstract. We compute the Riemann curvature tensor for these 3D space-times. As the Weyl tensor vanishes the Ricci tensor is the only relevant tensor for these spaces, for this reason we contract and produce the non-vanishing Ricci tensor components and the Ricci Scalar. Using the Lorentz group we may transform the Ricci tensor to its simplest form; this is necessary to begin the Cartan-Karlhede equivalence algorithm.

The coframe and its dual frame, and the connection coefficients

To begin we write down the coframe of one-forms, by defining $F_{\pm} = H \pm D$:

$$
m_{\mu} = \theta_{1\mu} = dr,
-n_{\mu} = \theta_{2\mu} = -\frac{1}{\sqrt{2}} (dt + F'_{\phi}d\phi),
-\ell_{\mu} = \theta_{3\mu} = -\frac{1}{\sqrt{2}} (dt + F''_{\phi}d\phi).
$$

Although we have not calculated the dual coframe, we may define it due to the properties with a dual frame basis,

$$
\tilde{e}^{\nu}_\alpha \theta^\alpha = \tilde{e}^{\nu}_\alpha e^\alpha_{\mu} dx^\mu = \delta^\nu_{\mu} dx^\mu = dx^\mu
$$

Notice that this takes the metric

$$
g_{\mu \nu} = -2\ell_{\mu}n_{\nu} + 2m_{\mu}m_{\nu}
$$

Thus $g(e_2) = -\theta^3$ and $g(e_3) = -\theta^2$, meaning that if one has a 2 or 3 index on the top of a tensor when we apply the metric to lower the index it becomes a 3 or 2 respectively and the term is multiplied by a negative, i.e.,

$$
A^2 \rightarrow -A_3, 
A^3 \rightarrow -A_2
$$

where $A$ is some arbitrary one-form relative to the coframe basis.

The connection coefficients relative to this particular coframe are simply:

$$
\Gamma_{112} = \Gamma_{113} = \Gamma_{332} = \Gamma_{223} = 0
$$

$$
\Gamma_{221} = \frac{F'_{\phi}}{2D}, \quad \Gamma_{331} = -\frac{F''_{\phi}}{2D}
$$

$$
\Gamma_{321} = -\frac{F'}{2D}, \quad \Gamma_{213} = \frac{H'}{2D}, \quad \Gamma_{231} = -\frac{D'}{2D}
$$

With these connection coefficients we may continue to calculate the curvature two-form by combining these with $\theta^\alpha$'s to make the connection one-forms $\omega^{\alpha}$. 

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\[ \omega^1_2 = \omega^1_{12} = -\frac{F'}{2D} \theta^2 + \frac{D'}{2D} \theta^3, \]

\[ \omega^1_3 = \omega^1_{13} = \frac{D'}{2D} \theta^2 + \frac{F'}{2D} \theta^3, \]

\[ -\omega^3_3 = \omega^3_{23} = \frac{H'}{2D} \theta^1. \]

Notice that \( \omega^2_2 \) is not needed as \( \omega^2_2 = -\omega^3_2 = \omega^3_{23} = -\omega^3_3 \).

THE RIEMANN AND RICCI TENSOR COMPONENTS

We may calculate the components of the Riemann tensor, using the second Cartan structure equation

\[ d\omega^\alpha_\beta + \omega^\alpha_\gamma \omega^\gamma_\beta = R^\alpha_\beta \]

where \( R^\alpha_\beta = R^\alpha_{\beta \gamma \delta} \theta^\gamma \wedge \theta^\delta \) is defined as the Curvature two-form. Taking the connection 1-forms above we apply the exterior derivative,

\[
\begin{align*}
    d\omega^1_2 &= \left[ -\left( \frac{F'}{2D} \right)' - \frac{F'_+ D'}{4D^2} + \frac{F'_+ H'}{4D^2} - \frac{F'_+ D'}{4D^2} \right] \theta^1 \wedge \theta^2 \\
    &\quad + \left[ \left( \frac{D'}{2D} \right)' \right. \\
    d\omega^1_3 &= \left[ \left( \frac{D'}{2D} \right)' - \frac{F'_+ D'}{4D^2} + \left( \frac{D'}{2D} \right)' \right. \\
    &\quad + \left. \left( \frac{D'}{2D} \right)' \right] \theta^1 \wedge \theta^3 \\
    d\omega^3_3 &= 0.
\end{align*}
\]

Alternatively all non-vanishing wedge products of the connection 1-forms yields:

\[
\begin{align*}
    \omega^1_2 \wedge \omega^1_2 &= \omega^1_{12} \wedge \omega^1_{23} = \frac{F'_+ H'}{4D^2} \theta^1 \wedge \theta^2 - \frac{D'H'}{4D^2} \theta^1 \wedge \theta^3 \\
    \omega^1_3 \wedge \omega^3_3 &= -\omega^1_{13} \wedge \omega^3_{23} = \frac{D'H'}{4D^2} \theta^1 \wedge \theta^2 + \frac{F'_+ H'}{4D^2} \theta^1 \wedge \theta^3 \\
    \omega^3_1 \wedge \omega^3_1 &= \omega^3_{12} \wedge \omega^3_{13} = \left[ -\frac{F'_+ D'}{4D^2} - \left( \frac{D'}{2D} \right)' \right] \theta^2 \wedge \theta^3.
\end{align*}
\]

Adding these sums together in the appropriate manner, the Riemann tensor components are read off of the coefficients of the basis \( \theta^\alpha \wedge \theta^\beta \).
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\[ R_{12} = -\frac{1}{2} \left[ \frac{F''}{D} - \left( \frac{H'}{D} \right)^2 - \frac{D'H'}{D^2} \right] \theta^1 \wedge \theta^2 + \frac{1}{4} \left[ \frac{2D''}{D} - \left( \frac{H'}{D} \right)^2 \right] \theta^1 \wedge \theta^3 \]

\[ R_{13} = \frac{1}{4} \left[ 2D'' - \left( \frac{H'}{D} \right)^2 \right] \theta^1 \wedge \theta^2 + \frac{1}{2} \left[ \frac{F''}{D} + \left( \frac{H'}{D} \right)^2 - \frac{D'H'}{D^2} \right] \theta^1 \wedge \theta^3 \]

\[ R_{33} = -\frac{1}{4} \left( \frac{H'}{D} \right)^2 \theta^2 \wedge \theta^3. \]

Thus the non-vanishing and algebraically independent components of the Riemann tensor, relative to the coframe, will be \( R_{1212}, R_{1213}, R_{1313} \) and \( R_{3233} \).

To build the Ricci tensor we contract along the first and third index of the Riemann tensor

\[ Ric_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta} \]

Computing \( (\alpha \beta) = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\} \) exhausts all algebraically independent components of the tensor, from this list the non-zero components are:

\[ Ric_{11} = -2R_{1213} = -\frac{1}{2} \left[ \frac{2D''}{D} - \left( \frac{H'}{D} \right)^2 \right] \]

\[ Ric_{22} = R_{1212} = -\frac{1}{2} \left[ \frac{F''}{D} - \left( \frac{H'}{D} \right)^2 - \frac{D'H'}{D^2} \right] \]

\[ Ric_{23} = R_{1213} - R_{3223} = \frac{1}{2} \left[ \frac{D''}{D} \right] \]

\[ Ric_{33} = R_{1313} = \frac{1}{2} \left[ \frac{F''}{D} + \left( \frac{H'}{D} \right)^2 - \frac{D'H'}{D^2} \right] \]

where we have lowered the upper-index in the Riemann tensor to exploit the anti-symmetry of the first pair of the indices. There is one more component that is relevant to the construction of the Riemann tensor, and it comes from contracting the Ricci tensor, \( R = Ric^\alpha_{\alpha} \) - the Ricci scalar:

\[ R = Ric_{11} - 2Ric_{23} = -\frac{1}{2} \left[ \frac{4D''}{D} - \left( \frac{H'}{D} \right)^2 \right] \]
To bring this in line with the quantities defined in [1, 2] we note

\[ \Phi_{00} = \frac{1}{2} \text{Ric}_{ab} e^a \ell^b = \text{Ric}_{22}. \]

\[ \Phi_{22} = \frac{1}{2} \text{Ric}_{ab} n^a n^b = \text{Ric}_{33}. \]

\[ \Phi_{10} = \frac{1}{\sqrt{2}} \text{Ric}_{ab} m^a \ell^b = \text{Ric}_{12}. \]

\[ \Phi_{12} = \frac{1}{\sqrt{2}} \text{Ric}_{ab} m^a n^b = \text{Ric}_{13}. \]

\[ \Phi_{11} = \frac{1}{6} (\text{Ric}_{ab} m^a n^b + \text{Ric}_{ab} n^a \ell^b) = \frac{1}{6} (\text{Ric}_{11} + \text{Ric}_{32}). \]

\[ \Lambda = \text{R} \]

For the Godel-like spacetimes we find that \( \Phi_{10} = \Phi_{12} = 0 \) and the non-zero components are:

\[ \Phi_{00} = -\frac{1}{4} \left[ \frac{F''}{D} - \left( \frac{H'}{D} \right)^2 - \frac{D'H'}{D^2} \right] \]

\[ \Phi_{22} = \frac{1}{4} \left[ \frac{F''}{D} + \left( \frac{H'}{D} \right)^2 - \frac{D'H'}{D^2} \right] \]

\[ \Phi_{11} = -\frac{1}{12} \left[ \frac{D''}{D} - \left( \frac{H'}{D} \right)^2 \right] \]

\[ \Lambda = -\frac{1}{2} \left[ \frac{4D''}{D} - \left( \frac{H'}{D} \right)^2 \right] \]

To normalize the components of the Ricci tensor we must use the Lorentz freedom to bring the components in line with the table in [1] on page 7. To summarize the frame freedoms we have a boost:

\( \tilde{\theta}^1 = \theta^1, \tilde{\theta}^2 = \sqrt{A} \theta^2, \tilde{\theta}^3 = \frac{1}{\sqrt{A}} \theta^3, \)

and null rotations about \( n^a \) and \( \ell^a \):

\( \tilde{\theta}^1 + B \theta^2, \tilde{\theta}^2 = \theta^2, \tilde{\theta}^3 = \theta^3 + B \theta^1 + \frac{B^2}{2} \theta^2, \)

\( \tilde{\theta}^1 + C \theta^3, \tilde{\theta}^2 = \theta^2 + C \theta^1 + \frac{C^2}{2} \theta^3, \tilde{\theta}^3 = \theta^3. \)

**Questions to Ask at the First Step of the C.K. Algorithm**

Prior to the canonical form, answer these questions:

1. Give me an argument why our null rotations about \( n^a \) and \( \ell^a \) have already been exhausted. That is, why does \( B = C = 0 \) for null rotations produce part of the ‘canonical’ form.

2. When is \( \Phi_{00} = \Phi_{22} \) prior to making a boost? Give a differential equation in terms of \( H \) and \( D \).

3. When does \( \Phi_{00} \) or \( \Phi_{22} \) vanish? If \( \Phi_{22} = 0 \) can we change our coframe so that \( \Phi_{00} \rightarrow \Phi_{22} \) in the new coframe?
(4) Assuming this is the case, given a parameter for the boost that will set \( \Phi_{22} = 1 \).

(5) When \( \Phi_{00} \neq \Phi_{22} \) what do we choose for the Lorentz boost parameter \( A \) so that in the new coframe \( \tilde{\Phi}_{00} = \tilde{\Phi}_{22} \)?

(6) When do both \( \Phi_{00} \) and \( \Phi_{22} \) both vanish so that boosts have no effect?

After we have used as much of the frame freedom as possible we will have two cases \( \Phi_{00} = \Phi_{22} \) or \( \Phi_{00} = 0 \) and \( \Phi_{22} = 1 \) and two more sets of questions to answer:

**Case 1:** \( \Phi_{00} = \Phi_{22} \).

(1) When \( \Phi_{00} = \Phi_{22} = 3\Phi_{11} \) with \( \Phi_{00} \neq 0 \)? What is the Segre type in this case?

(2) When \( \Phi_{00} = \Phi_{22} \neq 0 \) and \( \Phi_{11} \neq \Phi_{00} \)? What is the Segre type in this case?

(3) When \( \Phi_{00} = \Phi_{22} = 0 \) with \( \Phi_{11} \neq 0 \)? What is the Segre type in this case?

(4) When \( \Phi_{00} = \Phi_{22} = \Phi_{11} = 0 \)? What is the Segre type in this case?

**Case 2:** \( \Phi_{00} = 0, \ \Phi_{22} = 1 \).

(1) When \( \Phi_{11} \neq 0 \)? What is the Segre type in this case?

(2) When \( \Phi_{11} = 0 \)? What is the Segre type in this case?

**References**
