MATH 2030: ASSIGNMENT 5

Introduction to Eigenvalues and Eigenvectors

Q.1: pg 271 , q 6. Show that $\mathbf{v}$ is an eigenvector of A and find the corresponding eigenvalue, where $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

A.1. Matrix multiplication yields

$$A\mathbf{v} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}$$

As $\mathbf{v} \neq \mathbf{0}$ this vector is an eigenvector with eigenvalue $\lambda = 0$.

Q.2: pg 272 , q 8. Show that λ is an eigenvalue of A and find one eigenvector corresponding to this eigenvalue, where $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ and $\lambda = -2$.

A.2. To see if $\lambda = -2$ is an eigenvalue of the matrix A we row reduce the matrix with $A - (-2)I = A + 2I$

$$A + 2I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

this shows it is in fact an eigenvalue, because the null space of $A + 2I$ is non-trivial. By applying the same row reduction to $[A + 2I|0]$ one may show any solution to the equation $(A + 2I)\mathbf{x} = \mathbf{0}$ has the form

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad (1)$$

Q.3: pg 272, q 18. Find the eigenvalues and eigenvectors of A geometrically , where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, corresponding to a counterclockwise rotation of $\pi/4$ (90 degrees) about the origin.

A.3. In $\mathbb{R}^2$, the matrix A changes the angle $\theta_{\mathbf{v}}$ any vector $\mathbf{v}$ makes with the basis vector $\mathbf{e}_1^t = [1, 0]$ by $\theta_{\mathbf{v}} + \pi/4$ in radians. By definition an eigenvector is a vector whose direction, or in this case angle, is unchanged by the matrix-transformation. Thus there are no eigenvectors for the matrix representing a counterclockwise rotation of 90 degrees.

This is reflected in the fact that the characteristic equation for this matrix has no real-valued roots and hence no real-valued eigenvalues or eigenvectors.

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

The complex-valued roots are $\pm i$ where $i = \sqrt{-1}$. The eigenvectors will be complex valued as well.

Q.4: pg 273 , q 26. Find all eigenvalues of the matrix A. Give bases for each of the corresponding eigenspaces for the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

A.4. To determine the eigenvalues of A we compute the characteristic equation

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} &= (2-\lambda)^2 + 1 \\ &= 4 - 4\lambda + \lambda^2 + 1 \\ &= \lambda^2 - 4\lambda + 5 \end{aligned}$$

The roots in this case may be found using the quadratic formula, we find $\lambda = 2 \pm i$. We conclude there are no real eigenvalues or eigenvectors.

Q.5: pg 273 , q 35.

- Show that the eigenvalues of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are the solutions of the quadratic equation $\lambda^2 - tr(A)\lambda + detA = 0$, where the trace of A is tr(A).
- Show that the eigenvalues of the matrix A in the first part of this question are
$$\lambda = \frac{1}{2}(a + d \pm \sqrt{(a-d)^2 + 4bc}$$
- Show that the trace and determinant of the matrix A in the first part of the question are then
$$tr(A) = \lambda_1 + \lambda_2, \; det(A) = \lambda_1 \lambda_2$$
where λ_1 and λ_2 are eigenvalues of the matrix A.

A.5.

- Calculating the characteristic equation for this matrix, $det(A - \lambda I)$ we find the left hand side yields a polynomial
$$\begin{aligned} \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= (a-\lambda)(d-\lambda) - bc \\ &= ad - (a+d)\lambda + \lambda^2 - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - tr(A)\lambda + det(A). \end{aligned}$$
- Using the quadratic formula and denoting the solutions $\lambda_1 = \lambda_+$ and $\lambda_@ = \lambda_-$ we have
$$\begin{aligned} \lambda_\pm &= \frac{tr(A) \pm \sqrt{tr(A) - 4det(A)}}{2} \\ &= \frac{1}{2}(a + d \pm \sqrt{a^2 + 2ad + d^2 - 4(ad - bc)}) \\ &= \frac{1}{2}(a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}) \\ &= \frac{1}{2}(a + d \pm \sqrt{(a-d)^2 + 4bc}) \end{aligned}$$

- Adding $\lambda_1+\lambda_2$ the signs of the term under square root cancel, leaving $a+d$ which is the trace of A, tr(A). To verify the last identity,

$$\begin{aligned}\lambda_1\lambda_2 &= \frac{1}{4}(a+d+\sqrt{(a-d)^2+4bc})(a+d-\sqrt{(a-d)^2+4bc})\\ &= \frac{1}{4}[(a+d)^2-((a-d)^2+4bc)]\\ &= \frac{1}{4}[a^2+2ad+d^2-(a^2-2ad+d^2+4bc)=\frac{1}{4}(4ad-4bc)\end{aligned}$$

Canceling the 4 in the numerator and denominator, we are left with $det(A) = ad-bc$.

Determinants

Q.6: pg 292 q 2. Use the cofactor expansion along the first row and along the first column to calculate the determinant of the matrix

$$\begin{vmatrix}1 & 0 & -2\\ 3 & 3 & 2\\ 0 & -1 & 1\end{vmatrix}$$

A.6. The cofactor expansion along the first row is

$$\begin{aligned}det(A) &= 1\begin{vmatrix}3 & 2\\ -1 & 1\end{vmatrix} - 0\begin{vmatrix}3 & 2\\ 0 & 1\end{vmatrix} + (-2)\begin{vmatrix}3 & 3\\ 0 & -1\end{vmatrix}\\ &= (3+2)-2(-3)=11\end{aligned}$$

The cofactor expansion along the first column will yield the same value

$$\begin{aligned}det(A) &= 1\begin{vmatrix}3 & 2\\ -1 & 1\end{vmatrix} - 3\begin{vmatrix}0 & -2\\ -1 & 1\end{vmatrix} + 0\begin{vmatrix}0 & -2\\ 3 & 2\end{vmatrix}\\ &= (3+2)-3(-2)=11\end{aligned}$$

Q.7: pg 292 q 8. Use the cofactor expansion along any row or column that seems convenient for the matrix

$$\begin{vmatrix}1 & 2 & 3\\ -4 & 0 & 4\\ -3 & -2 & -1\end{vmatrix}$$

A.7. We perform the cofactor expansion along the second column,

$$\begin{aligned}det(A) &= -2\begin{vmatrix}-4 & 4\\ -3 & -1\end{vmatrix} + 0 - (-2)\begin{vmatrix}1 & 3\\ -4 & 4\end{vmatrix}\\ &= 2[-(4+12)+(4+12)]=0\end{aligned}$$

Q.8: pg 293 q 34. Using the properties of determinants, calculate the determinant for the given matrix by inspection. Explain your reasoning

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

A.8. Applying the row operation $R_3 - R_1$, then $R_3 - R_2$ leaves the determinant unchanged

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

One final row operation $R_4 + R_3$ produces an upper-triangular matrix whose determinant is easily computed

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Q.9: pg 293 q 46. Use the appropriate theorem to determine all values of k for which A is invertible, where $A = \begin{bmatrix} k & k & 0 \\ k^2 & 4 & k^2 \\ 0 & k & k \end{bmatrix}$.

A.9. Cofactor expanding along the first row we find

$$\begin{aligned} det(A) &= k\begin{vmatrix} 4 & k^2 \\ k & k \end{vmatrix} - k\begin{vmatrix} k^2 & k^2 \\ 0 & k \end{vmatrix} \\ &= k(4k - k^3) - k(k^3) \\ &= 4k^2 - 2k^4 \end{aligned}$$

Factoring this polynomial and equating it to zero we find $2k^2(k+\sqrt{2})(k-\sqrt{2}) = 0$ so that the roots are $k = 0, \pm\sqrt{2}$; these are the values that must be avoided as the determinant of A vanishes for these values and hence is not invertible.

Q.10: pg 292 q 54. If B is invertible, prove that $det(B^{-1}AB) = det(A)$.

A.10. Computing the left hand side, we use the fact that $det(AB) = det(A)det(B)$ twice and the fact that $det(B^{-1}) = \frac{1}{\det(A)}$:

$$det(B^{-1})det(A)det(B) = \frac{det(B)}{det(B)}det(A) = det(A)$$

References

[1] D. Poole, Linear Algebra: A modern introduction - 3rd Edition, Brooks/Cole (2012).