MATH 2030: ASSIGNMENT 6

EIGENVALUES AND EIGENVECTORS OF $n \times n$ MATRICES

Q.1: pg 309, q 2. For the given matrix,

$$A = \begin{bmatrix} 1 & -9 \\ 1 & -5 \end{bmatrix}$$

calculate

(1) The characteristic polynomial of $A$.
(2) The eigenvalues of $A$.
(3) A basis for each eigenspace of $A$.
(4) the algebraic and geometric multiplicity of each value

A.1.

(1) The characteristic polynomial of $A$ will be $\det(A - \lambda I)$:

$$\begin{vmatrix} 1 - \lambda & -9 \\ 1 & -5 - \lambda \end{vmatrix} = -(1 - \lambda)(5 + \lambda) + 9$$

expanding this we find the polynomial

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2.$$ 

(2) Equating this polynomial to zero, we find that the roots will be $\lambda = -2, -2$; this is the only value to satisfy $\det(A - \lambda I) = 0$, -2 is an eigenvalue with algebraic multiplicity 2.

(3) Computing the null space of the matrix $A + 2I = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}$ we find that a non-trivial solution to the homogeneous problem $(A + 2I)x = 0$ will satisfy $x_1 = -3x_2$. Thus the corresponding basis eigenvector for the eigenspace of the eigenvalue $\lambda = -2$ of $A$ is

$$x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(4) The eigenvalue $\lambda = -2$ has algebraic multiplicity 2 and geometric multiplicity 1.

Q.2: pg 309, q 10. For the given matrix,

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

calculate

(1) The characteristic polynomial of $A$.
(2) The eigenvalues of $A$. 

1
A basis for each eigenspace of $A$

the algebraic and geometric multiplicity of each value

(1) Taking the determinant of the matrix $A - \lambda I$ is easily done as this matrix is upper-triangular. The characteristic equation simply the product of the diagonals

$$det(A - \lambda I) = (2 - \lambda)(1 - \lambda)(3 - \lambda)(2 - \lambda).$$

(2) The eigenvalues of $A$ are then $\lambda = 2, 1, 3, 2$.

(3) Computing the null spaces of $A - 2I$, $A - I$ and $A - 3I$ we find the eigenspaces are spanned by the following vectors

$$E_1 = span \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 = span \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad E_3 = span \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

(4) For $\lambda = 1, 2, 3$ have algebraic multiplicity and geometric multiplicity both equal to 1 for each eigenvalue respectively.

Q.3: pg 310, q 13. Prove that if $A$ is invertible with eigenvalue $\lambda$ and corresponding eigenvector $x$, then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$ with corresponding eigenvector $x$.

A.3. If $x$ is an eigenvalue of $A$, with eigenvalue $\lambda$ then $Ax = \lambda x$. As $A$ is invertible, we may apply its inverse to both sides to get

$$x = \lambda I x = A^{-1} (\lambda x) = \lambda A^{-1} x$$

Multiplying by $1/\lambda$ on both sides show that $x$ is an eigenvector of $A^{-1}$ with $\lambda = \frac{1}{\lambda}$ since

$$A^{-1} x = \frac{1}{\lambda} x.$$ 

Q.4: pg 310, q 16. Suppose $A$ is a $3 \times 3$ matrix with eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = 1$ respectively. Find $A^{20} x$, if $x = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
A.4. We will give solutions for the vector given here and the vector given in the
text \( \mathbf{v}_b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \). It is easily shown that \( \mathbf{x} = \mathbf{v}_1 + \mathbf{v}_3 \), while \( \mathbf{v}_b = 1\mathbf{v}_1 - 1\mathbf{v}_2 + 2\mathbf{v}_3 \).

Computing \( A^{20}\mathbf{x} \) is then
\[
A^{20}\mathbf{x} = \left( - \frac{1}{3} \right)^{20} \mathbf{v}_1 + (1)^{20}\mathbf{v}_3 = \begin{bmatrix} 3^{-20} + 1 \\ 1 \\ 1 \end{bmatrix}
\]

while the vector \( \mathbf{v}_b \) yields
\[
A^{20}\mathbf{v}_b = \left( - \frac{1}{3} \right)^{20} \mathbf{v}_1 - \left( \frac{1}{3} \right)^{20} \mathbf{v}_2 + 2(1)^{20}\mathbf{v}_3 = \begin{bmatrix} -3^{-20} - 3^{-20} + 2 \\ -3^{-20} + 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\]

Q.5: pg 310, q 17. With \( \mathbf{v}_i \) and \( \lambda_i \) and \( \mathbf{x} \) as in the previous question, determine \( A^k\mathbf{x} \) for arbitrary \( k \).

A.5. Generalizing the result we find,
\[
A^k\mathbf{x} = \left( - \frac{1}{3} \right)^k \mathbf{v}_1 + (1)^k\mathbf{v}_3 = \begin{bmatrix} (-1)^{-k}3^{-k} + 1 \\ 1 \\ 1 \end{bmatrix}
\]

while the vector \( \mathbf{v}_b \) yields
\[
A^k\mathbf{v}_b = \left( - \frac{1}{3} \right)^k \mathbf{v}_1 - \left( \frac{1}{3} \right)^k \mathbf{v}_2 + 2(1)^k\mathbf{v}_3 - \begin{bmatrix} (-1)^{-k} - 13^{-k} + 2 \\ -3^{-k} + 2 \end{bmatrix}
\]

Q.6: pg 310, q 19.

- Show that for any square matrix \( A \), \( A^t \) and \( A \) have the same characteristic polynomial and hence the same eigenvalues.
- Give an example of a \( 2 \times 2 \) matrix \( A \) for which \( A^t \) and \( A \) have different eigenspaces.

A.6.

- Noting that \( det(A^t) = det(A) \) we examine the characteristic polynomial of \( A \) and use this fact, \( det(A - \lambda I) = det([A - \lambda I]^t) = det(A^t - \lambda I) = det(A^t - \lambda I) \). This shows the characteristic polynomials for \( A \) and its transpose are the same, and hence they have the same eigenvalues.

- Consider the matrix \( A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \) this has eigenvalues \( \lambda = 1, 2 \) with eigenspaces spanned by
\[
E_1 = span \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right), \quad E_2 = span \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).
\]

The matrix \( A^t \) has the eigenspaces
\[
E_1 = span \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right), \quad E_2 = span \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\]
Q.7: pg 310, q 22. If \( v \) is an eigenvector of \( A \) with corresponding eigenvalue \( \lambda \) and \( c \) a scalar, show that \( v \) is an eigenvector of \( A - cI \) with corresponding eigenvalue \( \lambda - c \).

A.7. Make the matrix \( A - cI \) and contract with the vector \( x \), one finds

\[
(A - cI)x = Ax - cx = \lambda x - cx = (\lambda - c)x.
\]

This proves that the vector \( x \) corresponding to \( \lambda \) the eigenvalue of \( A \) is an eigenvector corresponding to \( \lambda - c \) for the matrix \( A - cI \).

Q.8: pg 311, q 21. Let \( A \) be an idempotent matrix, meaning \( A^2 = A \). Show that \( \lambda = 0 \) or \( \lambda = 1 \) are the only possible eigenvalues of \( A \).

A.8. Suppose \( \lambda \) is any eigenvalue of \( A \) with corresponding eigenvector \( x \), then \( \lambda^2 \) will be an eigenvalue of the matrix \( A^2 \) with corresponding eigenvector \( x \). However, \( A^2 = A \) and so \( \lambda^2 = \lambda \) for the eigenvector \( x \). This can only occur if \( \lambda = 0 \) or 1.

Q.9: pg 310, q 23. For the matrix,

\[
A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}:
\]

- Find the eigenvalues and eigenspaces of this matrix.
- Using the appropriate theorem, and the previous example determine the eigenvalues and eigenspaces of \( A^{-1} \), \( A - 2I \) and \( A + 2I \).

A.9.

- This matrix has eigenvalues \( \lambda = -2, 5 \) with eigenspaces spanned by the following vectors respectively:

\[
E_{-2} = \text{span} \left( \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right), \quad E_5 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\]

- Using this result we see that the eigenvalues for \( A^{-1} \) are then \( \lambda = -\frac{1}{2}, \frac{1}{5} \) with eigenspaces

\[
E_{-\frac{1}{2}} = \text{span} \left( \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right), \quad E_{\frac{1}{5}} = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\]

- the eigenvalues and eigenspaces for \( A - 2I \) are \( \lambda = -4, 3 \) and

\[
E_0 = \text{span} \left( \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right), \quad E_3 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\]

- the eigenvalues and eigenspaces for \( A + 2I \) are \( \lambda = 0, 7 \) and

\[
E_{-4} = \text{span} \left( \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right), \quad E_7 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\]
Q.10: pg 311, q 39. Use the helpful fact:

**Proposition 0.1.** any square matrix $A$ that may be partitioned as $A = \begin{bmatrix} P & Q \\ O & S \end{bmatrix}$ where $P$ and $S$ are square matrices and $O$ is the zero matrix, then $\det A = (\det P)(\det S)$.

to prove that if a square matrix $A = \begin{bmatrix} P & Q \\ O & S \end{bmatrix}$ partitioned so that $P, S$ are square matrices then the characteristic polynomial of $A$ is

$$
c_A(\lambda) = c_P(\lambda)c_S(\lambda).
$$

A.10. Computing $A - \lambda I$ we may partition this new matrix using $I_{n-p}$ and $I_p$ where $p$ is the size of the matrix $P$,

$$
A - \lambda I = \begin{bmatrix} P - \lambda I_p & Q \\ O & S - \lambda I_{n-p} \end{bmatrix}
$$

Taking the determinant of this matrix we find $\det(A - \lambda I) = \det(P - \lambda I_p)\det(S - \lambda I_{n-p})$, noting that $c_A(\lambda) = \det(A - \lambda I)$ we find this last identity is exactly what was needed

$$
c_A(\lambda) = c_P(\lambda)c_S(\lambda).
$$

**References**